Bragg fiber

Introduction to Fiber Optics course

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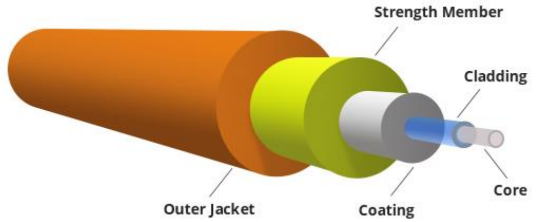
22/12/20

Motivation - Intro



- By far, most of the optical fibers in use today, depending on Total Internal Reflection for guiding light.
- Using this phenomenon as the physics basis for the fiber, force the need for core and cladding which have refractive indices.
- In turn the refractive indices, contribute to the propagation loss in the fiber:
  - Material absorption
  - Radiation loss – scattering, dispersion, etc.
- The possibility of other mechanism, was first pointed out by Yeh *et al [1]* - ***Bragg Fiber***

https://community.fs.com/blog/the-advantages-and-disadvantages-of-optical-fibers.html

Motivation – Bragg Fiber

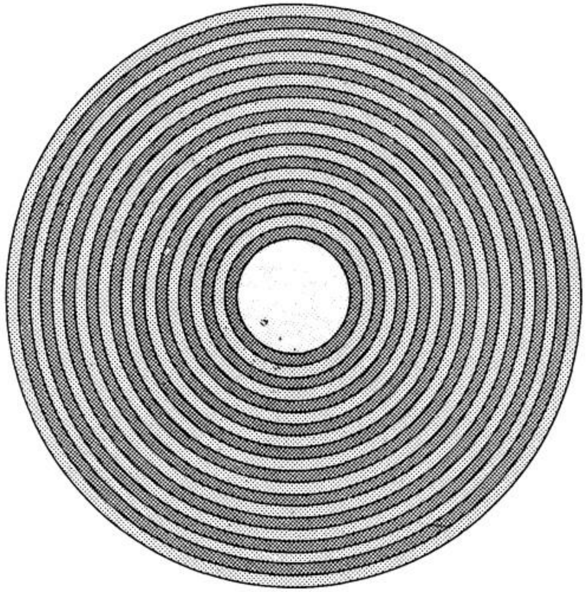
- ***Bragg fiber*** is leaning on Bragg reflection phenomenon in a cylindrical fiber to obtain lossless confined propagation in a core with a lower refractive index than that of the cladding medium.

- Because Bragg fiber and conventional fibers utilized different guiding mechanisms, we opening many possibilities difficult to achieve otherwise:

  - Use of low refractive index (even **air**) as the propagation medium, giving rise to low propagation losses, in terms of material absorption, scattering, dispersion, etc.

  - Bragg fiber modes are truly single mode (if chosen correctly)

  - Elimination of undesirable polarization dependent



Yeh, P., Yariv, A., & Maron, E. (1978). THEORY OF BRAGG FIBER. J Opt Soc Am, 68(9), 1196–1201.

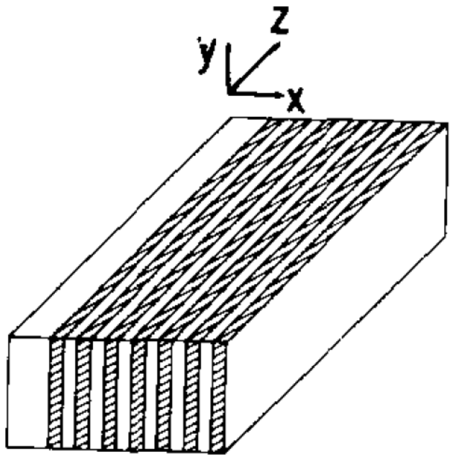
Bragg Reflection - Planar Waveguide

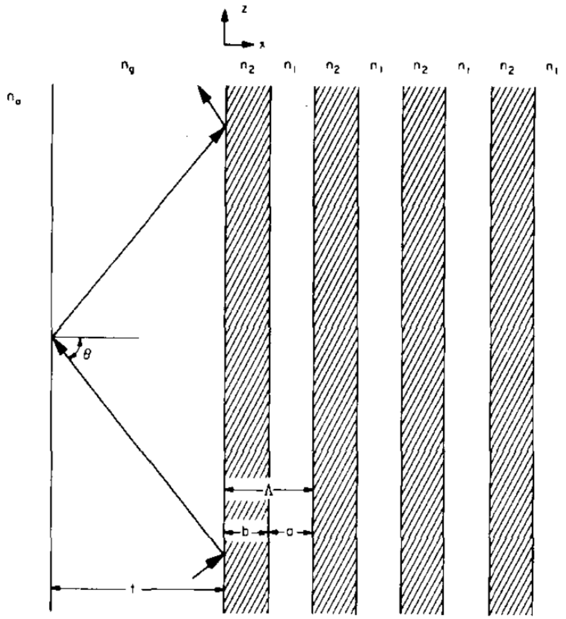
- Optical dielectric waveguides with slab configuration, like fibers, require the index of refraction of the inner layer to exceed that of the bounding media (identical to fibers)

- In contrast, Bragg configuration can support lossless propagation in a low index slab – providing that the bounding media are periodic.

- The periodic pattern constrain the modes (for most) in the slab, as we will show in the following analysis.

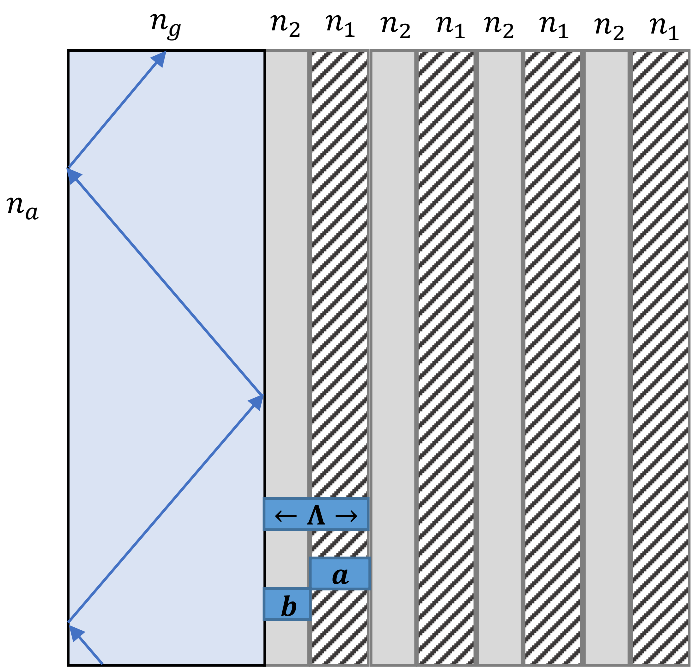
- We will start the analysis for **one side** Bragg formation, and then expand it for 2-side Bragg formation.





Yeh, P., & Yariv, A. (1976). Bragg reflection waveguides. Optics Communications, 19(3), 427-430.

Bragg Reflection Waveguide



- Let's consider the indices of refraction:

$$n_a < n_g < n_1, n_2$$

- Where n_a is the free space index, n_g is the slab index and n_1, n_2 are the periodic Bragg structure indices.

- To show the possibility of guiding modes in this structure, we will focus our discussion on **TE modes**. For this transverse mode the only field components are $\boldsymbol{E_y}, \boldsymbol{H_x}$ and $\boldsymbol{H_z}$.

- Each of this fields, say E_y satisfied the wave equation (with $e^{i\omega t}$ dependence):

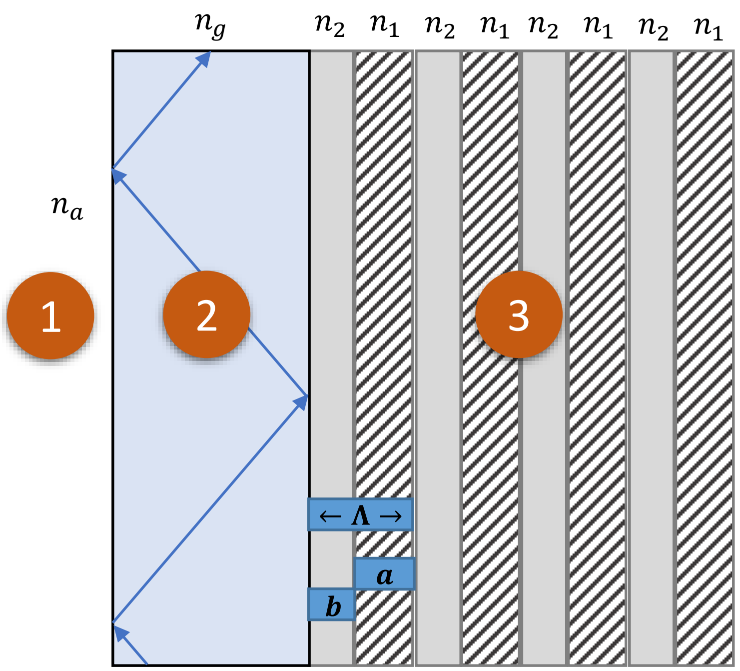
$$\nabla_t^2 \psi + \frac{\omega^2}{c^2} n^2(x)\psi = 0$$

(Wavefunction oscillates in time with a well-defined *constant* angular frequency ω)

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \frac{\omega^2}{c^2} n^2(x) E_y = 0$$

- E_y is a propagating wave in z direction, so: $E_y = E(x) \cdot e^{i\beta z}$

Bragg Reflection Waveguide



- The wave equation become:

$$\frac{\partial^2 E(x)}{\partial x^2} + \left(\frac{\omega^2}{c^2} n^2(x) - \beta^2\right) E(x) = 0$$

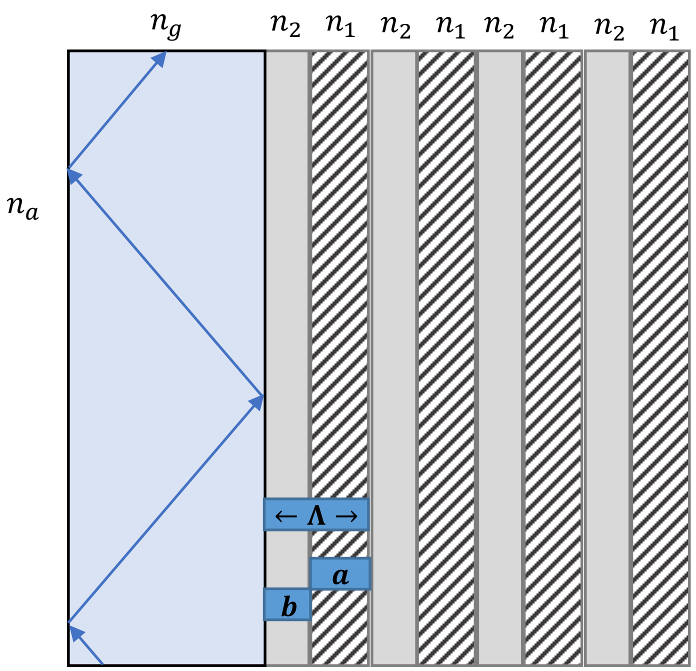
- We can choose our solution, as long it fits our derivative equation, so we take the solution in the form:

$$E(x) = \begin{cases} (1)\ \exp\big(q_a(x+t)\big) & x < -t \\ (2)\ c_1 \cos\big(k_g x\big) + c_2 \sin\big(k_g x\big) & -t \le x < 0 \\ (3)\ E_K(x) \cdot \exp(iKx) & x \ge 0 \end{cases}$$

- Where:

$$q_a = \sqrt{\beta^2 - \left(\frac{\omega}{c} n_a\right)^2} \quad , \quad k_g = \sqrt{\left(\frac{\omega}{c} n_g\right)^2 - \beta^2}$$

Bragg Reflection Waveguide



- Our solution compromise with

Conventional slab waveguide

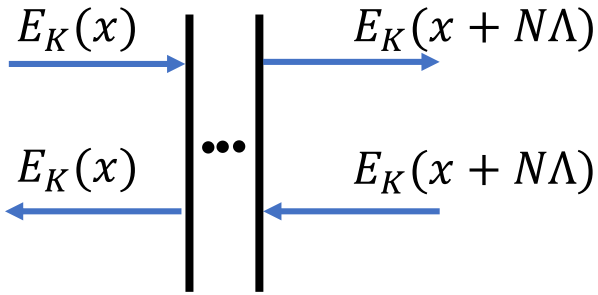
1) Evanesce wave - $q_a(x+t)$ *is negative imaginary*
2) Transverse wave
3) Wave define by the periodic pattern

- We will focus on equation - $(3)\ E_K(x) \cdot exp(iKx)$.

- According to Floquet's theorem this wave must have Bloch form, so that $E(x)$ is periodic with a period $\Lambda = a + b$.

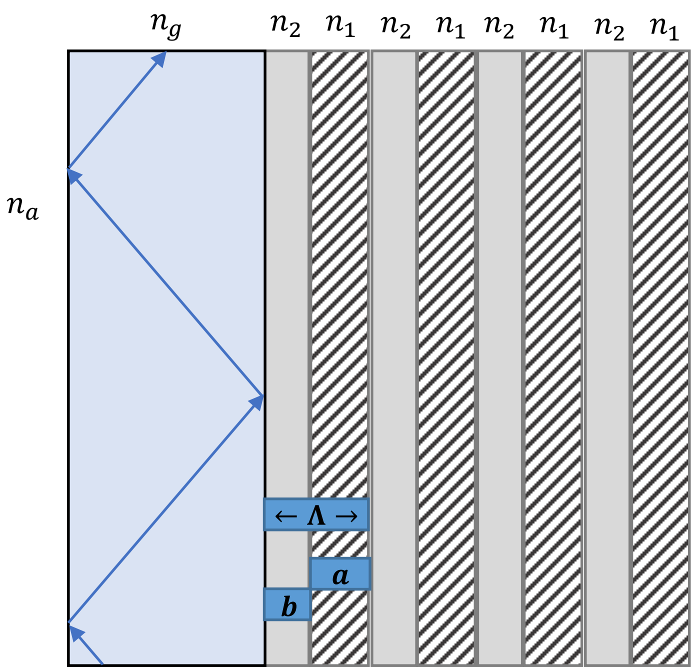
$$\boldsymbol{E_K(x+\Lambda) = E_K(x)}$$

- The field solution is obtained through the ***Transfer Matrix Method***, which link the complex amplitude of the incident and reflected wave from both side on the boundary.



$$\begin{bmatrix} E_K(x+N\Lambda)|_{incident} \\ E_K(x+N\Lambda)|_{reflect} \end{bmatrix} = T \cdot \begin{bmatrix} E_K(x)|_{incident} \\ E_K(x)|_{reflect} \end{bmatrix}$$

Bragg Reflection Waveguide



- The *Transfer Matrix Method* will give us the following variables:
  - $A = e^{-ik_{1x}a}\left[\cos(k_{2x}b) - \frac{i}{2}\left(\frac{k_{2x}}{k_{1x}} - \frac{k_{1x}}{k_{2x}}\right)\sin(k_{2x}b)\right]$
  - $B = e^{ik_{1x}a}\left[-\frac{i}{2}\left(\frac{k_{2x}}{k_{1x}} - \frac{k_{1x}}{k_{2x}}\right)\sin(k_{2x}b)\right]$
  - $C = B^*$; $D = A^*$

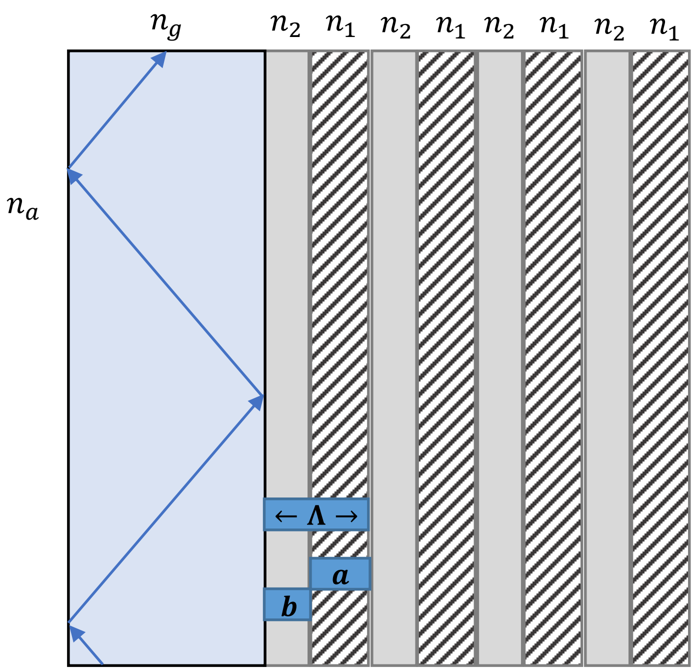
  $$where: k_{ix} = \sqrt{\left(\frac{\omega}{c}n_i\right)^2 - \beta^2} \quad , \quad i = 1,2$$

- **Note**: $AB - BC = 1$ (unimodal)

- After further development (that we will not go into here), the field $E(x)$ will equal to:

$$E(x) = E_K(x) \cdot e^{iKx}$$
$$= \left\{\left[a_0 e^{ik_{1x}(x-n\Lambda)} + b_0 e^{-ik_{1x}(x-n\Lambda)}\right] e^{-iK(x-n\Lambda)}\right\} \cdot e^{iKx}$$

Bragg Reflection Waveguide



- Where the coefficients a_0 and b_0 are:

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} B \\ e^{-iK\Lambda} - A \end{pmatrix}$$

- And the argument $e^{iK\Lambda}$ equal to:

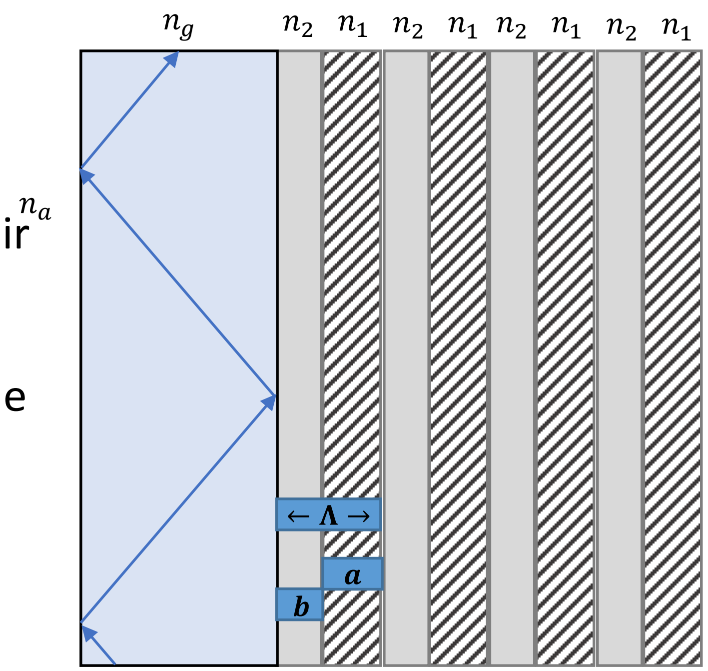
$$e^{iK\Lambda} = \left(\frac{A+D}{2}\right) \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1}$$

- In the regions where $\left(\frac{A+D}{2}\right)^2 < 1$,we get that $K = real$, which indicate on propagating Bloch waves

- Where $\left(\frac{A+D}{2}\right)^2 > 1$ the Bloch wavenumber become $K = m\pi/\Lambda + iK_i$, which contain an imaginary argument which in turn cause the **Bloch wave to evanescent**

- These are the so-called *"forbidden gaps"* of the periodic medium.

Bragg Reflection Waveguide



- To obtain solutions for the mode of the waveguide, we match the fields and their x derivatives, at the boundary of $x = 0$ and $x = t$.

- Using the solution for the field $E(x)$, the Bloch field $E_K(x)$ and the values of the coefficients a_0, b_0 we get the dispersion relation:

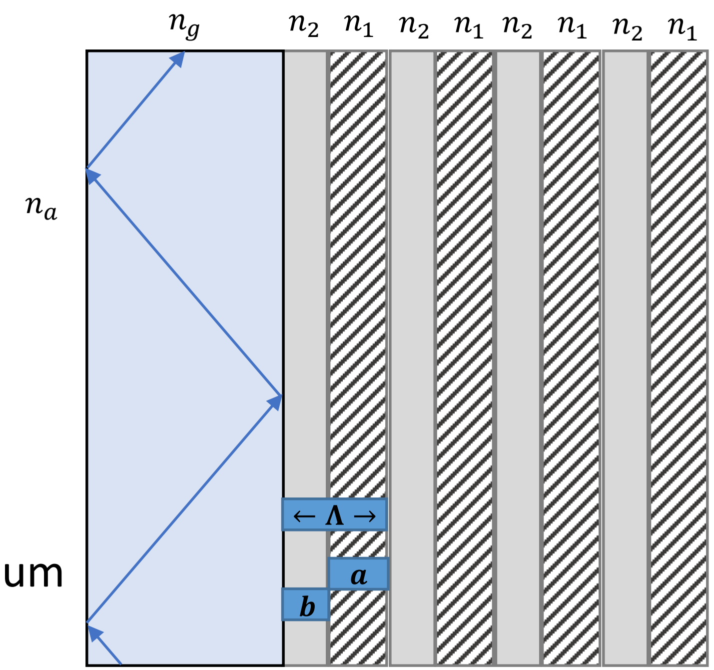
$$k_g \frac{q_a \cos(k_g t) - k_g \sin(k_g t)}{q_a \sin(k_g t) + k_g \cos(k_g t)} = -ik_{1x} \frac{e^{-iK\Lambda} - A - B}{e^{-iK\Lambda} - A + B}$$

Depend only on parameters of the guiding and substrate

Depend only on parameters of the periodic medium

- We interested in the evanescent Bloch wave.

Bragg Reflection Waveguide



$$k_g \frac{q_a \cos(k_g t) - k_g \sin(k_g t)}{q_a \sin(k_g t) + k_g \cos(k_g t)} = -ik_{1x} \frac{e^{-iK\Lambda} - A - B}{e^{-iK\Lambda} - A + B}$$

- For confined propagation β, q_a *and* k_g are real so that the left side of is a real number.
- The right side is real only when the propagating conditions in the periodic medium fall within one of the "forbidden gaps", meaning $\left(\frac{A+D}{2}\right)^2 > 1$.
- It follows that **<u>confined lossless modes</u>** of the composite waveguide exist.
- How to find the guiding modes?
  - solve for the eigenmode by starting with some value of $\beta < (\omega/c)n_g$.\
  - For a given ω, this determine the k_g, k_a, k_{1x}, k_{2x}
  - If the resulting values of A and D correspond to a "forbidden gap" $\left(\frac{A+D}{2}\right)^2 > 1 \quad \rightarrow$ the right side is (fixed) **real number.**
  - We then proceed to adjust the *thickness* of the guiding layer t until an equality results.

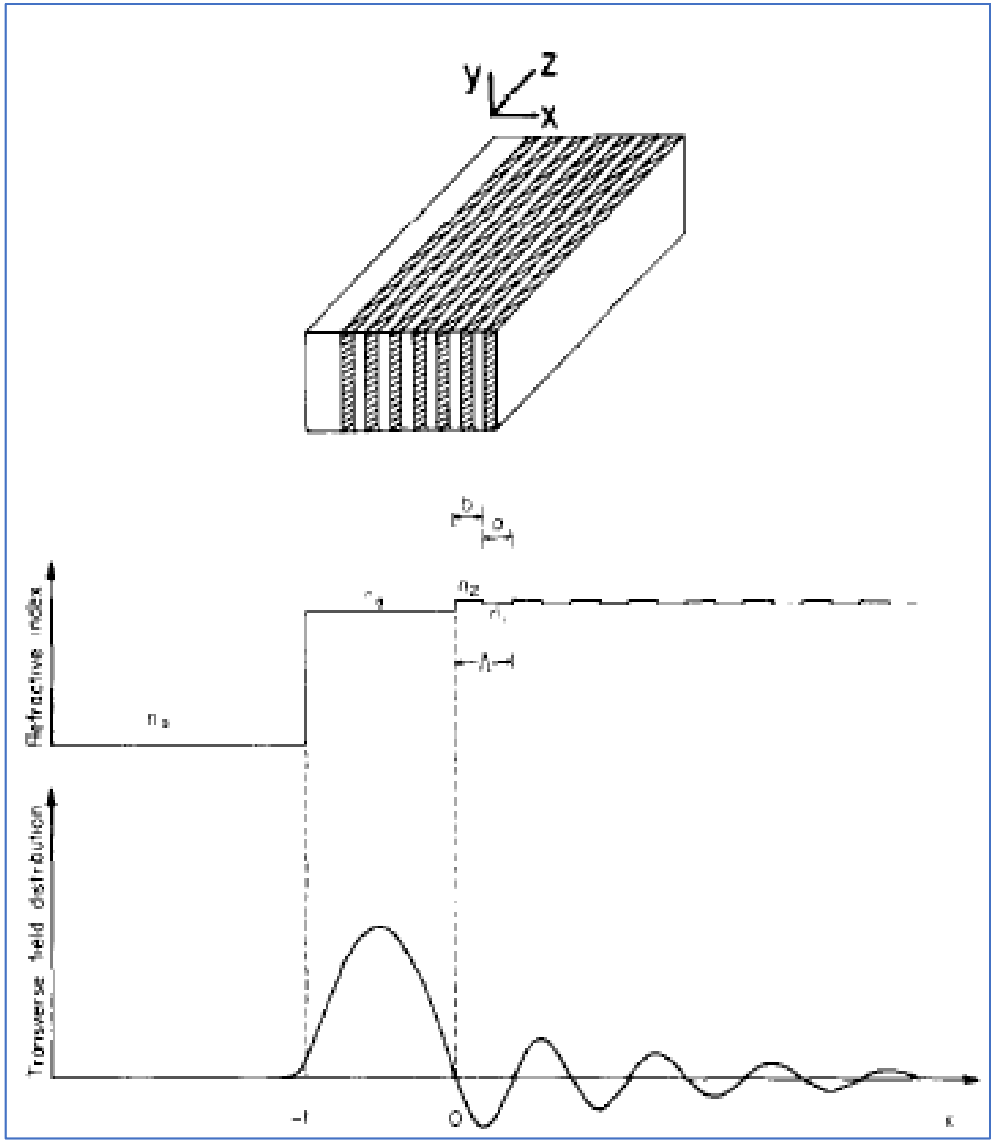
Bragg Reflection Waveguide

- A field distribution of such a waveguide is shown in the figure.

- We can see that in the periodic medium the field corresponds to a periodic pattern under an **evanescent envelope** $\boldsymbol{e^{-Kx}}$ as needed from a Bloch wave in a forbidden gap.

- The evanescent decay is nearly complete in several periods so that practical structures.

- In practical uses, few cells (say ten) are a good approximation to the semi-infinite layered medium assumed in the analysis.



Yeh, P., & Yariv, A. (1976). Bragg reflection waveguides. Optics Communications, 19(3), 427-430.

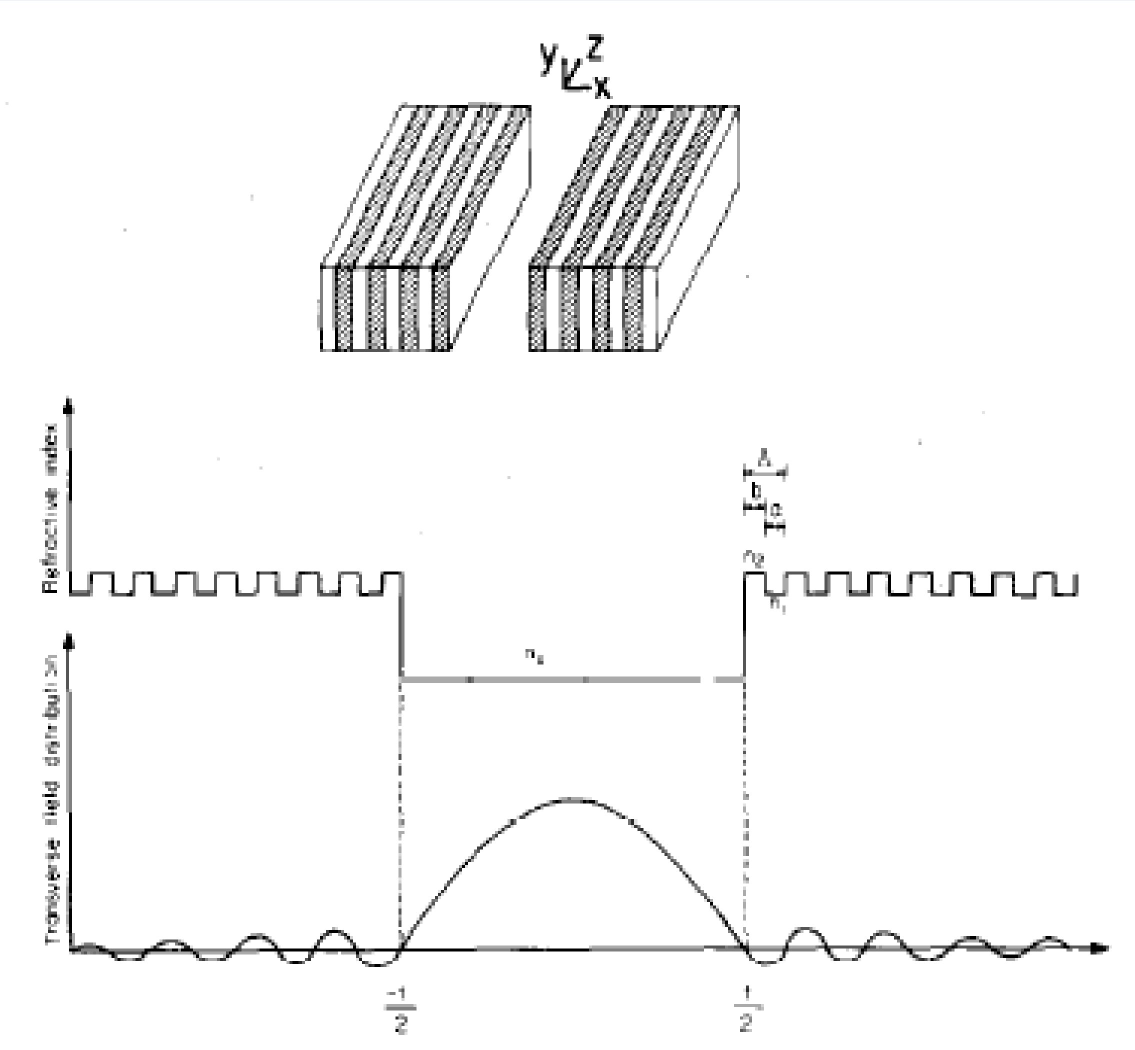
Bragg Reflection Waveguide

- Other possible architecture for the waveguide is double side periodic pattern.
- This will allow us to use air as the guiding medium.
- The equivalent equation for the mode (without analysis):

$$-ik_{1x}\frac{e^{-iK\Lambda}-A-B}{e^{-iK\Lambda}-A+B}=\begin{cases}k_a\tan(k_a t/2) & for\ even\ TE\ modes\\ k_a\cot(k_a t/2) & for\ odd\ TE\ modes\end{cases}$$

$$where\ k_a=\sqrt{\left(\frac{\omega}{c}n_a\right)^2-\beta^2}$$

- Because the existence of a given mode, requires the **simultaneous fulfillment** of the condition within the guiding layer and the Bragg condition in the layered media – the Bragg fiber display strong discrimination against higher modes.



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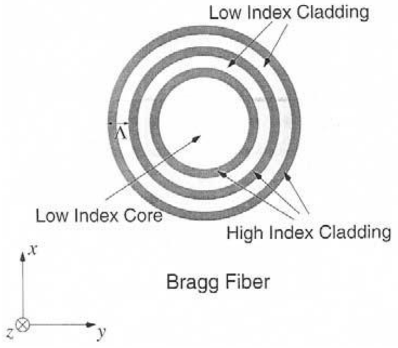
Bragg Fiber

- We want to utilized the properties of the lossless and selective frequency propagation in the Bragg slab, for an optical fiber.
- This will allow us to use different waveguide mechanism and overcome some of the limitation of the conventional fiber such as the **high core index** of refraction, and the necessity of **small core radius for single mode**.
- The mathematical approach for the fiber (compare to slab waveguide) will be different, because the geometrical difference between the 2 waveguides, prevent us from using Bloch Theorem (cartesian vs. cylinder symmetry).

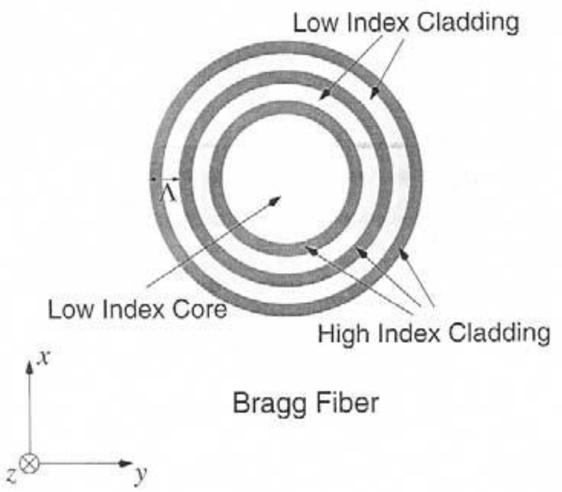


Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Bragg Fiber – How to Solve?

- Over the years many researchers solve this problem with different approaches, each one with its advantage and drawbacks. Every one of aim to find the guided modes in the Bragg fiber, with low restriction on the parameters and low calculation complexity

- The main approaches are:
  - FDTD (Finite-difference time-domain)
  - FEM (Finite element method)
  - Multiple scales approach
  - Asymptotic analysis for the cladding



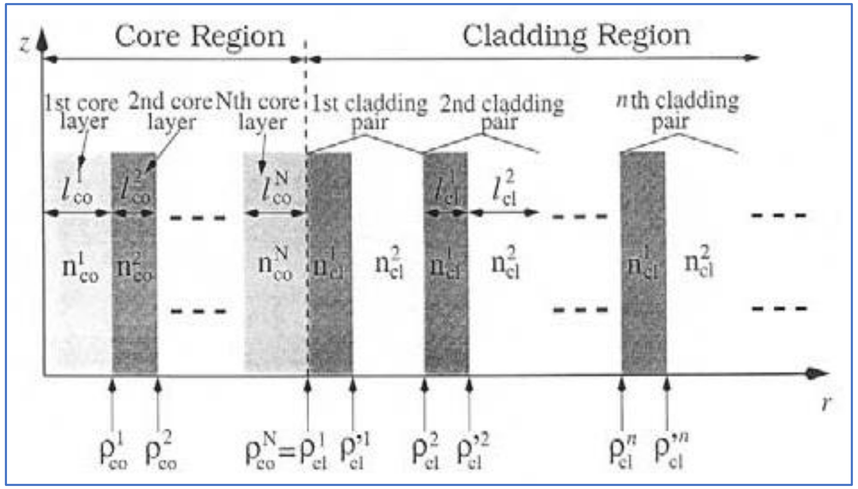
- We will focus our discussion on the ***Asymptotic Analysis Method*.**

Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Asymptotic Analysis Method

- In this formalism, an arbitrary part of the fiber (core, for example) is **treated exactly** (full analytical solution), and the other part is **approximate in the asymptotic** limit.

- Similarly, this approach divided to sub-methods, that differ by the part that calculate exactly and the part that is approximate.



- We will concentrate on the approach presented in the article – *"Asymptotic Matrix Theory of Bragg Fibers"*

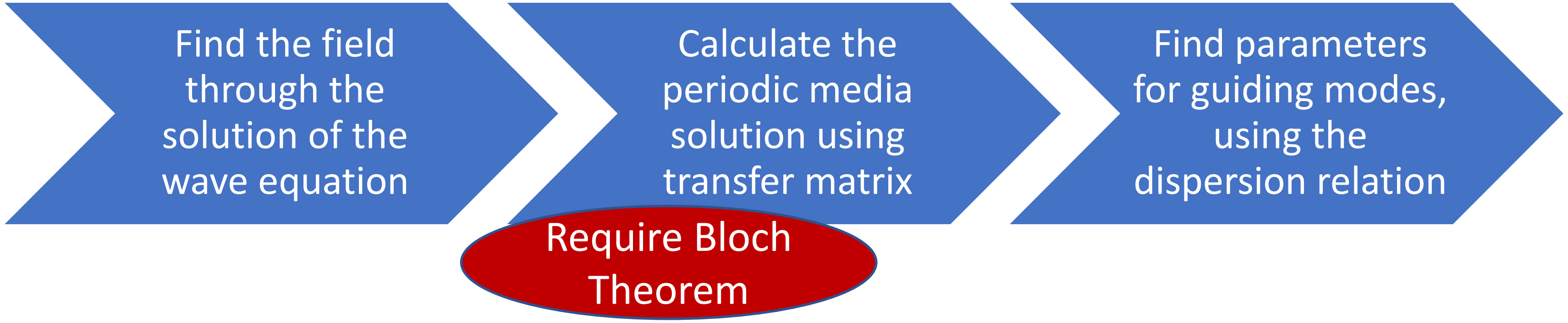
- In this article, both the core and cladding are made from number of layers, such that the core is treated exactly, and the cladding are treated at the asymptotic limit.

Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Asymptotic Analysis – Why?

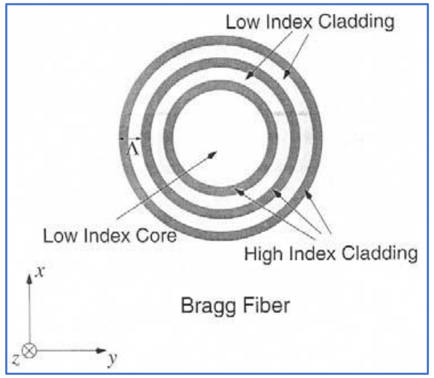
- For a **planar** air core Bragg waveguide the eigen solution that decays in the cladding structure can be found according to the Bloch theorem.

- For a **cylindrically** symmetric Bragg fiber, which is, strictly speaking, not periodic and for which the **Bloch theorem does not apply**, we cannot single out an eigen solution that decays in the fiber cladding layers.



- The problem was solved by using the asymptotic analysis, which enable us to find well approximate solution for the Bragg fiber.

Asymptotic Analysis – How?



- Exact Analytical solution is highly complex path, which slow down our ability to run fast and meaningful simulation or development.

- The main key for using asymptotic is that in the asymptotic limit, the exact solutions of Maxwell equations, which take the form of Bessel functions, can be approximated as:

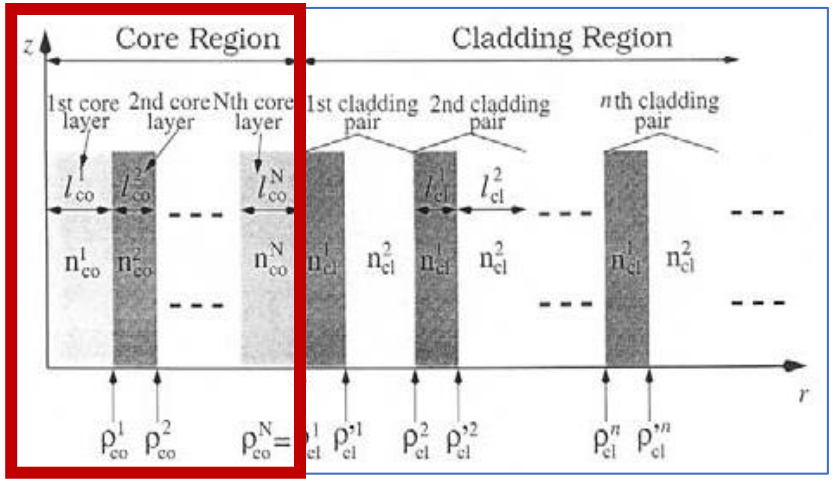
$$e^{-ikr}/\sqrt{r} \quad \boldsymbol{or} \quad e^{ikr}/\sqrt{r}$$

- And as we recall, in this form the solutions in Bragg fiber cladding resemble those in planar Bragg waveguides and eigen solutions in the fiber claddings can be similarly found – by comparison between the solution of the core (Bessel) and the cladding (Asymptotic) at the interface.

- <u>NOTICE</u>: One of the main goals of this article, compare to early articles, is to extend the analysis of the asymptotic formalism, in which the first **several dielectric layers** are treated exactly. The advantage is that we can choose the accuracy we want to get.

Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Solution in the Core Region



- The fiber core region consists of the first N concentric dielectric layers, which includes the **center low index core -** $\boldsymbol{n_{co}^{1}}$.
- The refractive index and thickness of layers in the core region can be chosen arbitrarily - $\boldsymbol{n_{co}^{i}}, \boldsymbol{l_{co}^{i}}$
- ρ is the distance from the center of the fiber:

$$Core:\ \rho_{co}^{i}, \qquad when\ i = 1,2,3, \dots, N$$

$$Cladding: \rho_{cl}^{i}, \qquad when\ i = 1,2,3, \dots, N$$

- As before the wave propagate in the z direction, as slowly changing function of time:

$$\boldsymbol{\psi(r, \theta, z, t) = \psi(r, \theta) e^{i(\beta z - \omega t)}}$$

- Where ψ can be every field of $E_{z/r/\theta}\ \ or\ \ H_{z/r/\theta}$.

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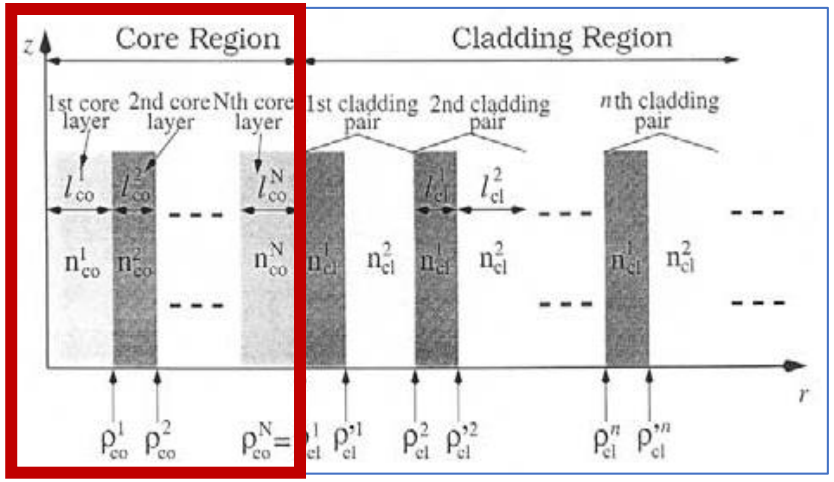
Solution in the Core Region



- The core is identical to conventional fiber (waveguide), so the transverse fields can be represented by E_z and H_z:

$$E_r = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(\frac{\partial}{\partial r}E_z + \frac{\omega\mu_0}{\beta}\frac{\partial}{r\partial\theta}H_z\right)$$

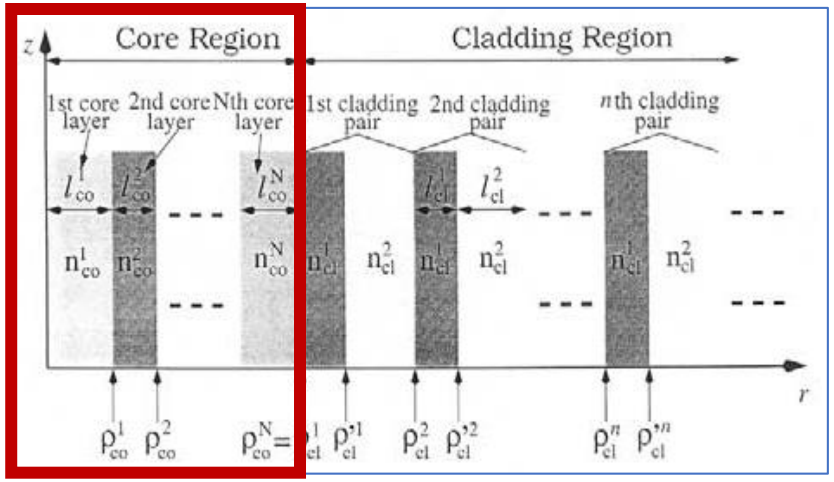
$$E_\theta = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(-\frac{\omega\mu_0}{\beta}\frac{\partial}{\partial r}H_z + \frac{\partial}{r\partial\theta}E_z\right)$$

$$H_r = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(\frac{\partial}{\partial r}H_z - \frac{\omega\epsilon_0 n^2}{\beta}\frac{\partial}{r\partial\theta}E_z\right)$$

$$H_\theta = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(\frac{\omega\epsilon_0 n^2}{\beta}\frac{\partial}{\partial r}E_z + \frac{\partial}{r\partial\theta}H_z\right)$$

- Where $\boldsymbol{n}$ is the index of refraction of the medium, β is the propagation constant, ω is the angular frequency, and ϵ_0 & μ_0 are the permittivity and permeability of the free space, respectively.

Solution in the Core Region

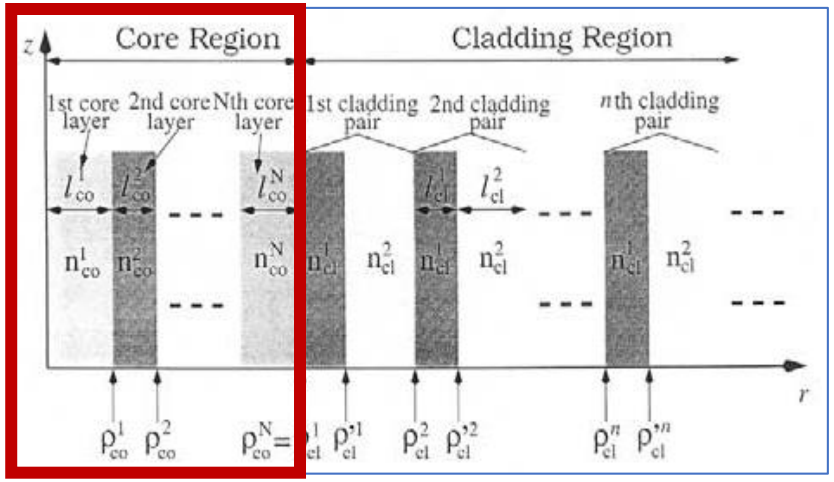


- Due to the cylindrical symmetry of Bragg fibers, we can take the azimuthal dependence of the field components as $\cos(l\theta)$
- For each l, the general solutions for E_z and H_z (*as we learn in class*) are the superposition of the Bessel functions – either $[\, J_l(x)\ \&\ Y_l(x)\,]$ or $[I_l(x)\ \&\ K_l(x)]$.
- In the **core medium** - the solutions are given by $[\, J_l(x)\ \&\ Y_l(x)\,]$, due to the real value of

$$k = \sqrt{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}$$

- Now we can write the solution of the transverse fields as a matrix using the Bessel functions

Solution in the Core Region



We get:

$$\begin{bmatrix} E_z \\ \frac{1}{i\beta}H_\theta \\ H_z \\ \frac{1}{i\beta}E_\theta \end{bmatrix} = M(n_{co}^i, k_{co}^i, r)\begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix}$$

General Solution of E_z and H_z

When the matrix M is defined by:

$$M(n_{co}^i, k_{co}^i, r) = \begin{bmatrix} J_l(k_{co}^i r) & Y_l(k_{co}^i r) & 0 & 0 \\ \frac{\omega\epsilon_0 (n_{co}^i)^2}{k_{co}^i\beta}J_l'(k_{co}^i r) & \frac{\omega\epsilon_0 (n_{co}^i)^2}{k_{co}^i\beta}Y_l'(k_{co}^i r) & \frac{l}{(k_{co}^i)^2 r}J_l(k_{co}^i r) & \frac{l}{(k_{co}^i)^2 r}Y_l(k_{co}^i r) \\ 0 & 0 & J_l(k_{co}^i r) & Y_l(k_{co}^i r) \\ \frac{l}{(k_{co}^i)^2 r}J_l(k_{co}^i r) & \frac{l}{(k_{co}^i)^2 r}Y_l(k_{co}^i r) & \frac{\omega\mu_0}{k_{co}^i\beta}J_l'(k_{co}^i r) & \frac{\omega\mu_0}{k_{co}^i\beta}Y_l'(k_{co}^i r) \end{bmatrix}$$

Where: A_i, B_i, C_i *and* D_i are constant within the ith layer ; and $k_{co}^i = \sqrt{\left(\frac{n_{co}^i\omega}{c}\right)^2 - \beta^2}$

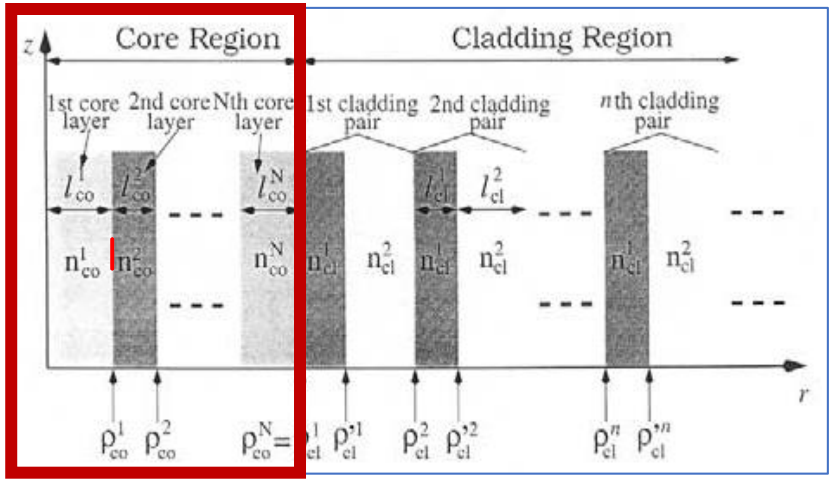
$$E_r = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(\frac{\partial}{\partial r}E_z + \frac{\omega\mu_0}{\beta}\frac{\partial}{r\partial\theta}H_z\right)$$

$$E_\theta = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(-\frac{\omega\mu_0}{\beta}\frac{\partial}{\partial r}H_z + \frac{\partial}{r\partial\theta}E_z\right)$$

$$H_r = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(\frac{\partial}{\partial r}H_z - \frac{\omega\epsilon_0 n^2}{\beta}\frac{\partial}{r\partial\theta}E_z\right)$$

$$H_\theta = \frac{i\beta}{\left(\frac{\omega^2}{c^2}\right)n^2 - \beta^2}\left(\frac{\omega\epsilon_0 n^2}{\beta}\frac{\partial}{\partial r}E_z + \frac{\partial}{r\partial\theta}H_z\right)$$

Solution in the Core Region



- As we defined at the start, the core is comprised from multiple layer, and until now we found the fields only in the first layer

- To find the fields in the $(i+1)$th layer, we apply the continuous conditions at the interface between 2 layers at $r = \rho_{co}^{i}$:

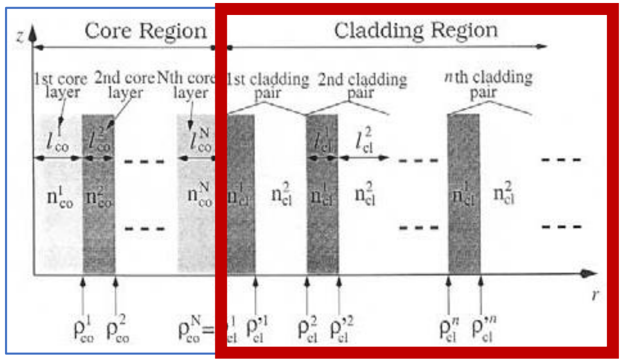
$$M\left(n_{co}^{i}, k_{co}^{i}, \rho_{co}^{i}\right)\begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix} = M\left(n_{co}^{i+1}, k_{co}^{i+1}, \rho_{co}^{i+1}\right)\begin{bmatrix} A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \end{bmatrix}$$

$$\begin{bmatrix} A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \end{bmatrix} = \underbrace{\left[M\left(n_{co}^{i+1}, k_{co}^{i+1}, \rho_{co}^{i+1}\right)\right]^{-1} M\left(n_{co}^{i}, k_{co}^{i}, \rho_{co}^{i}\right)}\begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix}$$

T_i - Transfer Matrix

- **<u>NOTICE</u>: In the first core layer, the coefficients B_1 and D_1 are zero, because $Y_l(x)$ is infinite at $x = 0$**

Solution in the Cladding Region



- In the cladding region, we will utilize the asymptotic approximation.
- The cladding consist of two types of **alternating** dielectric layers:
  - Type 1: refractive index n_{cl}^1 and thickness l_{cl}^1
  - Type 2: refractive index n_{cl}^2 and thickness l_{cl}^2
- As we discussed earlier, the asymptotic approximation relate the Bessel function to exponent function, as follow:

$$J(x) \sim \frac{a}{\sqrt{x}} \cos(x-b) \quad ; \quad Y(x) \sim \frac{a}{\sqrt{x}} \sin(x-b)$$

$$\psi\big(\alpha(x-\beta)\big) = \tilde{A} \cdot J\big(\alpha(x-\beta)\big) + \tilde{B} \cdot Y\big(\alpha(x-\beta)\big) = \left[Ae^{i\alpha(x-\beta)} + Be^{-i\alpha(x-\beta)}\right]/\sqrt{\alpha x}$$

- Meaning, that the fields that in an exact calculation equal to superposition of Bessel function can be approximate to superposition of $[e^{+i}, e^{-i}]$

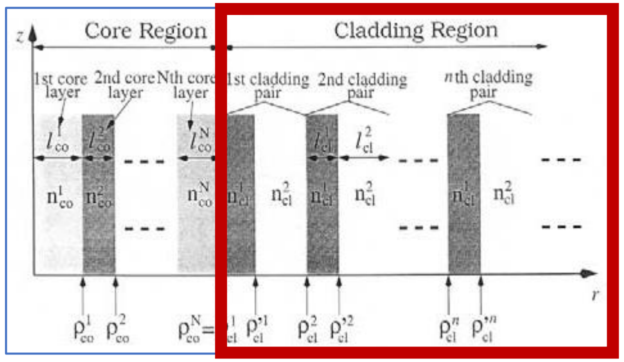
As $x \to 0$, $J_n(x)$ with $n \geqslant 0$ is finite. All others are infinite at the origin. Asymptotic forms of the solutions as $x \to \infty$ are

$$\begin{aligned}
J_n(x) &\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\
Y_n(x) &\sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\
H_n^{(1)}(x) &\sim \sqrt{\frac{2}{\pi x}} \exp\left[i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right] \\
H_n^{(2)}(x) &\sim \sqrt{\frac{2}{\pi x}} \exp\left[-i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]
\end{aligned} \tag{7-72}$$

J_n, Y_n, $H_n^{(1)}$, and $H_n^{(2)}$ all obey the same recursion relations [(7-52) through (7-55)]. Some functions related to Bessel functions are listed below.

De Heer, J. (1965). Mathematical Methods of Physics. By Jon Mathews and Robert L. Walker. Inorganic Chemistry, 4(1), 134.

Solution in the Cladding Region



- We found that the exact solution and the approximated form of E_z is:

$$E_z = A_i \cdot J_l\left(k_{cl}^i r\right) + B_i \cdot Y\left(k_{cl}^i r\right)$$

$$E_z = \begin{cases} \frac{1}{\sqrt{k_{cl}^1 r}}\left(a_n e^{ik_{cl}^1\left(r-\rho_{cl}^n\right)} + b_n e^{-ik_{cl}^1\left(r-\rho_{cl}^n\right)}\right) & \rho_{cl}^n < r < \rho_{cl}^n + l_{cl}^1 \\ \frac{1}{\sqrt{k_{cl}^2 r}}\left(a_n' e^{ik_{cl}^2\left(r-{\rho'}_{cl}^n\right)} + b_n' e^{-ik_{cl}^2\left(r-{\rho'}_{cl}^n\right)}\right) & {\rho'}_{cl}^n < r < {\rho'}_{cl}^n + l_{cl}^2 \end{cases}$$

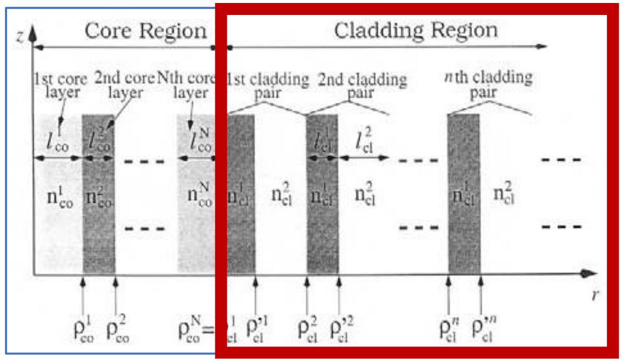
- **Its important to note**, that for this approximation to work, r must be large enough.

- This is one of the main advantage to use multiple layers as the core – we can choose to calculate as many layer as needed of the core as exact, and when we far enough from the axis, we can use the approximations.

- Same go for H_z (with c_n, c_n', d_n and d_n').

- With E_z & H_z we can calculate all the other field as before.

Solution in the Cladding Region

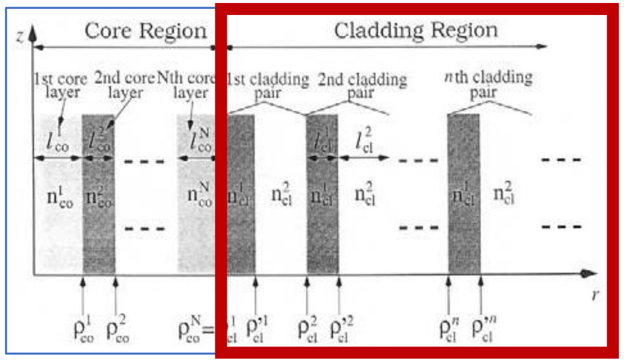


- To summarized our solution for the fields so far:

$$\rho_{cl}^{n} < r < \rho_{cl}^{n} + l_{cl}^{1}: \begin{cases} E_z = \dfrac{f_{TM}}{\sqrt{k_{cl}^{1} r}}\left(a_n e^{ik_{cl}^{1}(r-\rho_{cl}^{n})} + b_n e^{-ik_{cl}^{1}(r-\rho_{cl}^{n})}\right) \\ H_\theta = -\dfrac{\omega\epsilon_0 \left(n_{cl}^{1}\right)^2}{k_{cl}^{1}} \dfrac{f_{TM}}{\sqrt{k_{cl}^{1} r}}\left(a_n e^{ik_{cl}^{1}(r-\rho_{cl}^{n})} - b_n e^{-ik_{cl}^{1}(r-\rho_{cl}^{n})}\right) \\ H_z = \dfrac{f_{TE}}{\sqrt{k_{cl}^{1} r}}\left(c_n e^{ik_{cl}^{1}(r-\rho_{cl}^{n})} + d_n e^{-ik_{cl}^{1}(r-\rho_{cl}^{n})}\right) \\ E_\theta = -\dfrac{\omega\mu_0}{k_{cl}^{1}} \dfrac{f_{TE}}{\sqrt{k_{cl}^{1} r}}\left(c_n e^{ik_{cl}^{1}(r-\rho_{cl}^{n})} - d_n e^{-ik_{cl}^{1}(r-\rho_{cl}^{n})}\right) \end{cases}$$

$$\rho'^{n}_{cl} < r < \rho'^{n}_{cl} + l_{cl}^{2}: \begin{cases} E_z = \dfrac{f_{TM}}{\sqrt{k_{cl}^{2} r}}\left(a'_n e^{ik_{cl}^{2}(r-\rho'^{n}_{cl})} + b'_n e^{-ik_{cl}^{1}(r-\rho'^{n}_{cl})}\right) \\ H_\theta = -\dfrac{\omega\epsilon_0 \left(n_{cl}^{2}\right)^2}{k_{cl}^{2}} \dfrac{f_{TM}}{\sqrt{k_{cl}^{2} r}}\left(a'_n e^{ik_{cl}^{2}(r-\rho'^{n}_{cl})} - b'_n e^{-ik_{cl}^{2}(r-\rho'^{n}_{cl})}\right) \\ H_z = \dfrac{f_{TE}}{\sqrt{k_{cl}^{2} r}}\left(c'_n e^{ik_{cl}^{2}(r-\rho'^{n}_{cl})} + d'_n e^{-ik_{cl}^{2}(r-\rho'^{n}_{cl})}\right) \\ E_\theta = -\dfrac{\omega\mu_0}{k_{cl}^{2}} \dfrac{f_{Te}}{\sqrt{k_{cl}^{2} r}}\left(c'_n e^{ik_{cl}^{2}(r-\rho'^{n}_{cl})} - d'_n e^{-ik_{cl}^{2}(r-\rho'^{n}_{cl})}\right) \end{cases}$$

Solution in the Cladding Region



- **It should be noted** that the TM component (including E_z and H_θ) and the TE component (including E_θ and H_z) are decoupled in the asymptotic limit, with the TM component amplitude being f_{TM} and the TE component amplitude being f_{TE}.

- The solutions take form of the traveling wave ($e^{\pm i\beta z}$) with $1/\sqrt{r}$ - which mean that properties of the cylindrically symmetric Bragg stacks resemble those of **planar Bragg stacks**.

- Hence, the **fields at neighbor cladding pairs are the same** except an overall amplitude change of amplitude, which is direct consequence of **Bloch theorem**

- By matching fields at interfaces between dielectric layers, we can find the coefficients a_n, b_n, c_n, d_n, in the form of:

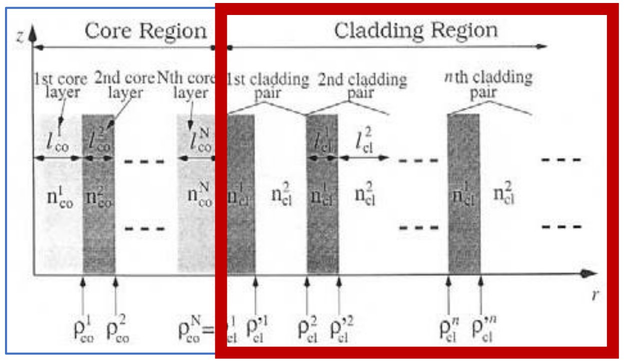
$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = (\lambda_{TM})^{n-1} \begin{bmatrix} B_{TM} \\ \lambda_{TM} - A_{TM} \end{bmatrix}$$

$$\begin{bmatrix} c_n \\ d_n \end{bmatrix} = (\lambda_{TE})^{n-1} \begin{bmatrix} B_{TE} \\ \lambda_{TE} - A_{TE} \end{bmatrix}$$

Change of Amplitude

Base Coefficients

Solution in the Cladding Region



- By matching fields at interfaces between dielectric layers, we can find the coefficients a_n, b_n, c_n, d_n:

$$A_{TE} = e^{ik_{cl}^1 l_{cl}^1}\left[i\frac{\left(k_{cl}^1\right)^2 + \left(k_{cl}^2\right)^2}{2k_{cl}^1 k_{cl}^2} \cdot \sin\left(k_{cl}^2 l_{cl}^2\right) + \cos\left(k_{cl}^2 l_{cl}^2\right)\right]$$

$$B_{TE} = ie^{-ik_{cl}^1 l_{cl}^1}\frac{\left(k_{cl}^1\right)^2 - \left(k_{cl}^2\right)^2}{2k_{cl}^1 k_{cl}^2}\sin\left(k_{cl}^2 l_{cl}^2\right)$$

$$A_{TM} = e^{ik_{cl}^1 l_{cl}^1}\left[i\frac{\left(n_{cl}^2\right)^4\left(k_{cl}^1\right)^2 + \left(n_{cl}^1\right)^4\left(k_{cl}^2\right)^2}{2(n_{cl}^1)^2(n_{cl}^2)^2 k_{cl}^1 k_{cl}^2} \cdot \sin\left(k_{cl}^2 l_{cl}^2\right) + \cos\left(k_{cl}^2 l_{cl}^2\right)\right]$$

$$B_{TE} = ie^{-ik_{cl}^1 l_{cl}^1}\frac{\left(n_{cl}^2\right)^4\left(k_{cl}^1\right)^2 + \left(n_{cl}^1\right)^4\left(k_{cl}^2\right)^2}{2(n_{cl}^1)^2(n_{cl}^2)^2 k_{cl}^1 k_{cl}^2}\sin\left(k_{cl}^2 l_{cl}^2\right)$$

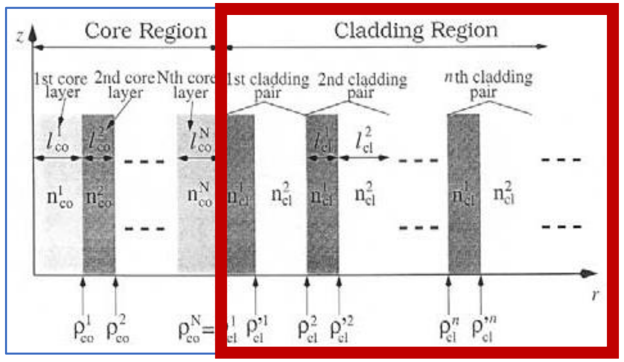
$$\boldsymbol{\lambda_{TE} = Re(A_{TE}) \pm \sqrt{[Re(A_{TE})]^2 - 1}}$$

$$\boldsymbol{\lambda_{TM} = Re(A_{TM}) \pm \sqrt{[Re(A_{TM})]^2 - 1}}$$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = (\lambda_{TM})^{n-1}\begin{bmatrix} B_{TM} \\ \lambda_{TM} - A_{TM} \end{bmatrix}$$

$$\begin{bmatrix} c_n \\ d_n \end{bmatrix} = (\lambda_{TE})^{n-1}\begin{bmatrix} B_{TE} \\ \lambda_{TE} - A_{TE} \end{bmatrix}$$

Solution in the Cladding Region



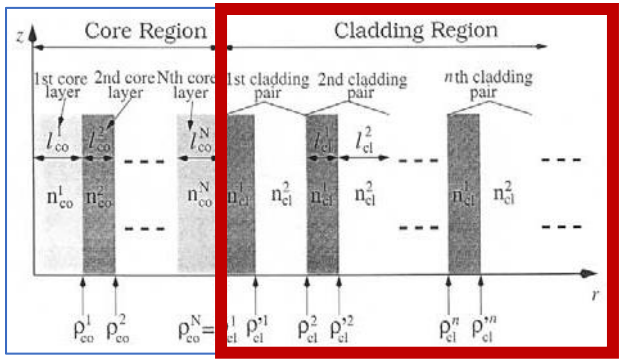
- The field amplitudes in type 2 layer of the $\boldsymbol{n}$th cladding pair can be found by applying the condition of E_z, E_θ, H_z and H_θ being continuous at $r = \rho'^{n}_{cl}$, which gives:

$$\begin{bmatrix} a'_n \\ b'_n \end{bmatrix} = \frac{1}{2}\sqrt{\frac{k_{cl}^2}{k_{cl}^1}} \begin{bmatrix} \left(1 + \frac{\left(n_{cl}^1\right)^2 k_{cl}^2}{(n_{cl}^2)^2 k_{cl}^1}\right) e^{ik_{cl}^1 l_{cl}^1} & \left(1 - \frac{\left(n_{cl}^1\right)^2 k_{cl}^2}{(n_{cl}^2)^2 k_{cl}^1}\right) e^{-ik_{cl}^1 l_{cl}^1} \\ \left(1 - \frac{\left(n_{cl}^1\right)^2 k_{cl}^2}{(n_{cl}^2)^2 k_{cl}^1}\right) e^{ik_{cl}^1 l_{cl}^1} & \left(1 + \frac{\left(n_{cl}^1\right)^2 k_{cl}^2}{(n_{cl}^2)^2 k_{cl}^1}\right) e^{-ik_{cl}^1 l_{cl}^1} \end{bmatrix} \cdot \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} c'_n \\ d'_n \end{bmatrix} = \frac{1}{2}\sqrt{\frac{k_{cl}^2}{k_{cl}^1}} \begin{bmatrix} \left(1 + \frac{k_{cl}^2}{k_{cl}^1}\right) e^{ik_{cl}^1 l_{cl}^1} & \left(1 - \frac{k_{cl}^2}{k_{cl}^1}\right) e^{-ik_{cl}^1 l_{cl}^1} \\ \left(1 - \frac{k_{cl}^2}{k_{cl}^1}\right) e^{ik_{cl}^1 l_{cl}^1} & \left(1 + \frac{k_{cl}^2}{k_{cl}^1}\right) e^{-ik_{cl}^1 l_{cl}^1} \end{bmatrix} \cdot \begin{bmatrix} c_n \\ d_n \end{bmatrix}$$

- As we can see, the transfer matrix depend on the wavenumbers and the thickness of the first layer.
  (the method propagate the wavefunction through the l_{cl}^1 , and then transfer through the boundary – change in the k of the medium)

Solutions for the Guided Modes



- The guided modes in a Bragg fiber are founded by matching the exact solution in the last core layer with the asymptotic solution in the first cladding layer at the interface $r = \rho_{co}^{N} = \rho_{cl}^{1}$:

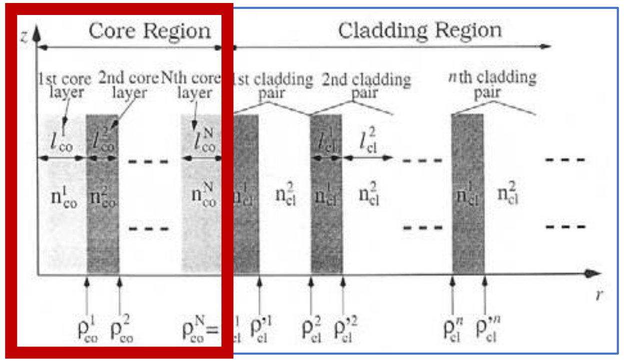
$$M(n_{co}^{N}, k_{co}^{N}, \rho_{co}^{N})\begin{bmatrix} A_N \\ B_N \\ C_N \\ D_N \end{bmatrix} = \begin{bmatrix} \frac{f_{TM}}{\sqrt{k_{cl}^{1}\rho_{cl}^{1}}}(\lambda_{TM} - A_{TM} + B_{TM}) \\ \frac{i\omega\epsilon_0 (n_{cl}^{1})^2}{k_{cl}^{1}\beta}\frac{f_{TM}}{\sqrt{k_{cl}^{1}\rho_{cl}^{1}}}(\lambda_{TM} - A_{TM} - B_{TM}) \\ \frac{f_{TE}}{\sqrt{k_{cl}^{1}\rho_{cl}^{1}}}(\lambda_{TE} - A_{TE} + B_{TE}) \\ \frac{i\omega\mu_0}{k_{cl}^{1}\beta}\frac{f_{TE}}{\sqrt{k_{cl}^{1}\rho_{cl}^{1}}}(\lambda_{TE} - A_{TE} - B_{TE}) \end{bmatrix}$$

Transfer Matrix

Constants in the Nth layer

Fields Solution at the first cladding layer

Solutions for the Guided Modes



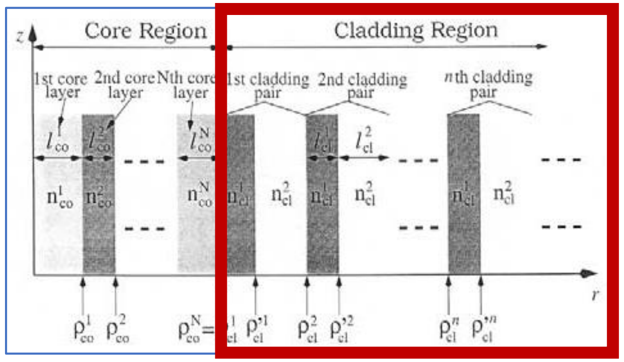
- We want to relate the coefficients of the Nth core layer, to the first core layer.
- As discussed before, in the **first** core layer - $B_1 = D_1 = 0$ (because $Y(x)$ is infinite at $x = 0$). We then denote $A_1\ as\ \mathcal{A}_{TM}$ and $C_1\ as\ \mathcal{C}_{TE}$:

$$\begin{bmatrix} A_N \\ B_N \\ C_N \\ D_N \end{bmatrix} = \boldsymbol{T_{N-1}} \cdots \boldsymbol{T_2} [M(n_{co}^2, k_{co}^2, \rho_{co}^1)]^{-1} \cdot \begin{bmatrix} J_l(k_{co}^1 \rho_{co}^1) & 0 \\ \frac{\omega\epsilon_0 (n_{co}^1)^2}{k_{co}^1 \beta} J_l'(k_{co}^1 \rho_{co}^1) & \frac{l}{(k_{co}^1)^2 \rho_{co}^1} J_l(k_{co}^1 r) \\ 0 & J_l(k_{co}^1 \rho_{co}^1) \\ \frac{l}{\left(k_{co}^i\right)^2 \rho_{co}^1} J_l(k_{co}^1 \rho_{co}^1) & \frac{\omega\mu_0}{k_{co}^i \beta} J_l'(k_{co}^1 \rho_{co}^1) \end{bmatrix} \cdot \begin{bmatrix} \mathcal{A}_{TM} \\ \mathcal{C}_{TE} \end{bmatrix}$$

- We will define the **full transfer matrix** as combination of all the transfer matrices between each boundaries are:

$$\boldsymbol{T} = [M(n_{co}^2, k_{co}^2, \rho_{co}^1) M^{-1}(n_{co}^2, k_{co}^2, \rho_{co}^2)] \cdots [M(n_{co}^N, k_{co}^N, \rho_{co}^{N-1}) M^{-1}(n_{co}^N, k_{co}^N, \rho_{co}^N)] =$$
$$\prod_{i=2}^{N} \left[M\left(n_{co}^i, k_{co}^i, \rho_{co}^{i-1}\right) M^{-1}\left(n_{co}^i, k_{co}^i, \rho_{co}^i\right)\right] = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}$$

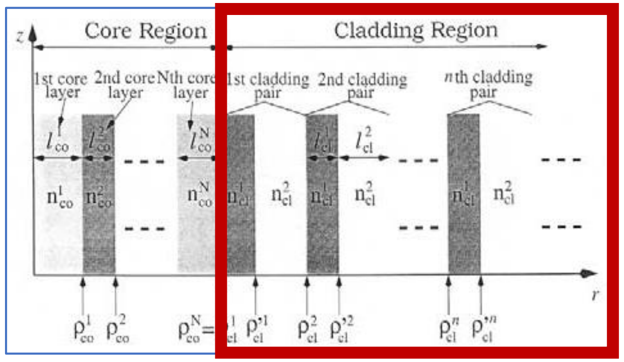
Solutions for the Guided Modes



- Substituting the argument into the equation, and implementing $\boldsymbol{T}$ into it, we get:

$$\begin{bmatrix} J_l(k_{co}^1\rho_{co}^1) & 0 \\ \dfrac{\omega\epsilon_0(n_{co}^1)^2}{k_{co}^1\beta}J_l'(k_{co}^1\rho_{co}^1) & \dfrac{l}{(k_{co}^1)^2\rho_{co}^1}J_l(k_{co}^1 r) \\ 0 & J_l(k_{co}^1\rho_{co}^1) \\ \dfrac{l}{\left(k_{co}^i\right)^2\rho_{co}^1}J_l(k_{co}^1\rho_{co}^1) & \dfrac{\omega\mu_0}{k_{co}^i\beta}J_l'(k_{co}^1\rho_{co}^1) \end{bmatrix} \cdot \begin{bmatrix} \mathcal{A}_{TM} \\ \mathcal{C}_{TE} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \dfrac{f_{TM}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TM} - A_{TM} + B_{TM}) \\ \dfrac{i\omega\epsilon_0\left(n_{cl}^1\right)^2}{k_{cl}^1\beta}\dfrac{f_{TM}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TM} - A_{TM} - B_{TM}) \\ \dfrac{f_{TE}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TE} - A_{TE} + B_{TE}) \\ \dfrac{i\omega\mu_0}{k_{cl}^1\beta}\dfrac{f_{TE}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TE} - A_{TE} - B_{TE}) \end{bmatrix}$$

Solutions for the Guided Modes



- From the last equation, we can see that $\mathcal{A}_{TM}$ and $\mathcal{C}_{TE}$ are linearly related to the field in the first cladding layer – f_{TM} and f_{TE} via a 4x4 transfer matrix $\boldsymbol{T}$.

- So, we have 4 equations with 4 independent variables, which is suffice to determine the propagation constant β and the field distribution of all guided Bragg fiber modes

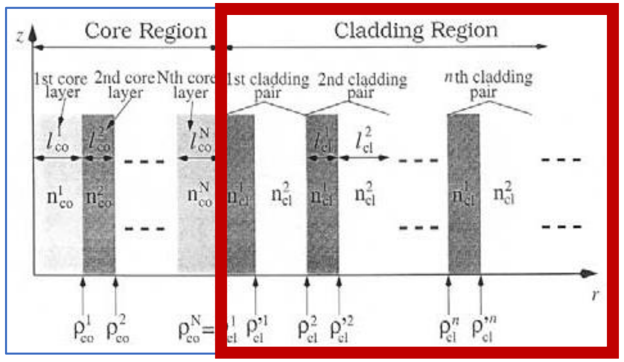
- For simplification we introduce 8 new parameters:

$$g_{TE}^{j} = t_{j3}(\lambda_{TE} - A_{TE} + B_{TE}) - \frac{i\omega\mu_0}{k_{cl}^1\beta} t_{j4}(\lambda_{TE} - A_{TE} - B_{TE}) \quad / \quad j = 1, \dots, 4$$

$$g_{TM}^{j} = t_{j1}(\lambda_{TM} - A_{TM} + B_{TM}) - \frac{i\omega\epsilon_0\left(n_{cl}^1\right)^2}{k_{cl}^1\beta} t_{j2}(\lambda_{TM} - A_{TM} - B_{TM}) \quad / \quad j = 1, \dots, 4$$

- Where t are the elements in the transfer matrix $\boldsymbol{T}$.

Solutions for the Guided Modes



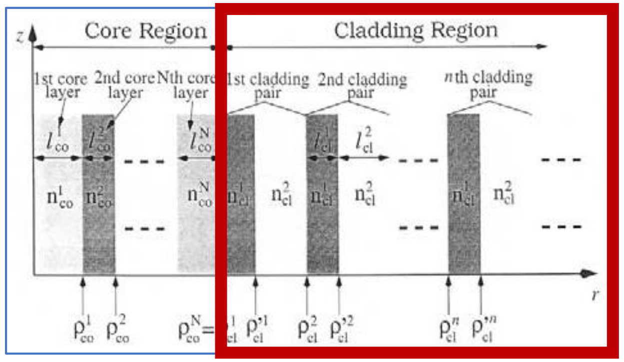
- These parameters will allow us to **rewrite and split** the previous equation between the last core layer and the first cladding layer, to get:

$$\begin{bmatrix} J_l(k_{co}^1\rho_{co}^1) & 0 \\ \frac{\omega\epsilon_0(n_{co}^1)^2}{k_{co}^1\beta}J_l'(k_{co}^1\rho_{co}^1) & \frac{l}{(k_{co}^1)^2\rho_{co}^1}J_l(k_{co}^1 r) \end{bmatrix}\cdot\begin{bmatrix}\mathcal{A}_{TM}\\ \mathcal{C}_{TE}\end{bmatrix} = \frac{1}{\sqrt{k_{cl}^1\rho_{cl}^1}}\begin{bmatrix} g_{TM}^1 & g_{TE}^1 \\ g_{TM}^2 & g_{TE}^2\end{bmatrix}\begin{bmatrix} f_{TM} \\ f_{TE}\end{bmatrix}$$

$$\begin{bmatrix} 0 & J_l(k_{co}^1\rho_{co}^1) \\ \frac{l}{\left(k_{co}^i\right)^2\rho_{co}^1}J_l(k_{co}^1\rho_{co}^1) & \frac{\omega\mu_0}{k_{co}^i\beta}J_l'(k_{co}^1\rho_{co}^1) \end{bmatrix}\cdot\begin{bmatrix}\mathcal{A}_{TM}\\ \mathcal{C}_{TE}\end{bmatrix} = \frac{1}{\sqrt{k_{cl}^1\rho_{cl}^1}}\begin{bmatrix} g_{TM}^3 & g_{TE}^3 \\ g_{TM}^4 & g_{TE}^4\end{bmatrix}\begin{bmatrix} f_{TM} \\ f_{TE}\end{bmatrix}$$

$$\begin{bmatrix} J_l(k_{co}^1\rho_{co}^1) & 0 \\ \frac{\omega\epsilon_0(n_{co}^1)^2}{k_{co}^1\beta}J_l'(k_{co}^1\rho_{co}^1) & \frac{l}{(k_{co}^1)^2\rho_{co}^1}J_l(k_{co}^1 r) \\ 0 & J_l(k_{co}^1\rho_{co}^1) \\ \frac{l}{\left(k_{co}^i\right)^2\rho_{co}^1}J_l(k_{co}^1\rho_{co}^1) & \frac{\omega\mu_0}{k_{co}^i\beta}J_l'(k_{co}^1\rho_{co}^1) \end{bmatrix}\cdot\begin{bmatrix}\mathcal{A}_{TM}\\ \mathcal{C}_{TE}\end{bmatrix} = \boldsymbol{T}\begin{bmatrix} \frac{f_{TM}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TM}-A_{TM}+B_{TM}) \\ \frac{i\omega\epsilon_0\left(n_{cl}^1\right)^2}{k_{cl}^1\beta}\frac{f_{TM}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TM}-A_{TM}-B_{TM}) \\ \frac{f_{TE}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TE}-A_{TE}+B_{TE}) \\ \frac{i\omega\mu_0}{k_{cl}^1\beta}\frac{f_{TE}}{\sqrt{k_{cl}^1\rho_{cl}^1}}(\lambda_{TE}-A_{TE}-B_{TE}) \end{bmatrix}$$

Solutions for the Guided Modes



- We will focus on the TE and TM modes – where $l = 0$.
  - The matrix $M(n_{co}^i, k_{co}^i, r)$ become block diagonalized to a two 2x2 matrices.

$$M(n_{co}^i, k_{co}^i, r) = \begin{bmatrix} J_l(k_{co}^i r) & Y_l(k_{co}^i r) & 0 & 0 \\ \frac{\omega\epsilon_0 (n_{co}^i)^2}{k_{co}^i \beta} J_l'(k_{co}^i r) & \frac{\omega\epsilon_0 (n_{co}^i)^2}{k_{co}^i \beta} Y_l'(k_{co}^i r) & 0 & 0 \\ 0 & 0 & J_l(k_{co}^i r) & Y_l(k_{co}^i r) \\ 0 & 0 & \frac{\omega\mu_0}{k_{co}^i \beta} J_l'(k_{co}^i r) & \frac{\omega\mu_0}{k_{co}^i \beta} Y_l'(k_{co}^i r) \end{bmatrix}$$

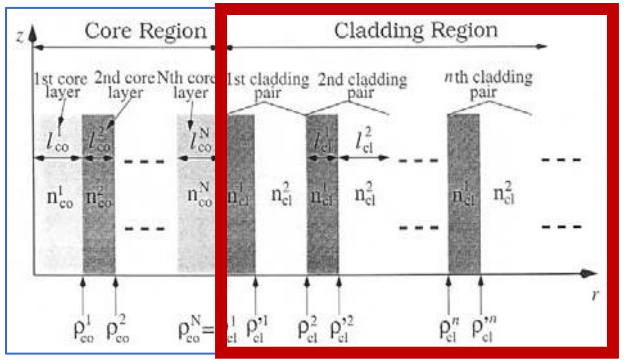
  - As a result, the transfer matrix $\boldsymbol{T}$ is also block diagonalized:

$$T = \begin{bmatrix} t_{11} & t_{12} & 0 & 0 \\ t_{21} & t_{22} & 0 & 0 \\ 0 & 0 & t_{33} & t_{34} \\ 0 & 0 & t_{43} & t_{44} \end{bmatrix}$$

  - And the part of the new parameters $g_{TE/TM}$ equal zero as well

$$g_{TM}^3 = g_{TM}^4 = g_{TE}^1 = g_{TE}^2 = 0$$

Solutions for the Guided Modes



- By the definition of TM mode, the H_z field much remain zero in the entire Bragg fiber, which demand:

$$\mathcal{C}_{TE} = 0 \ \ , f_{TE} = 0$$

- Using all this conditions the first split equation, we can get:

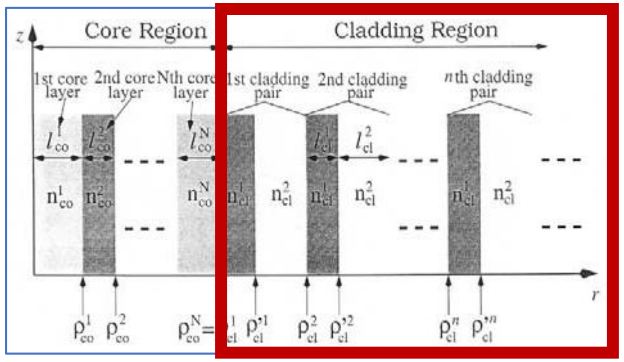
$$\begin{bmatrix} J_l(k_{co}^1 \rho_{co}^1) & 0 \\ \frac{\omega \epsilon_0 (n_{co}^1)^2}{k_{co}^1 \beta} J_l'(k_{co}^1 \rho_{co}^1) & \frac{l}{(k_{co}^1)^2 \rho_{co}^1} J_l(k_{co}^1 r) \end{bmatrix} \cdot \begin{bmatrix} \mathcal{A}_{TM} \\ \cancel{\mathcal{C}_{TE}} \end{bmatrix} = \frac{1}{\sqrt{k_{cl}^1 \rho_{cl}^1}} \begin{bmatrix} g_{TM}^1 & \cancel{g_{TE}^1} \\ g_{TM}^2 & \cancel{g_{TE}^2} \end{bmatrix} \begin{bmatrix} f_{TM} \\ \cancel{f_{TE}} \end{bmatrix}$$

$$\frac{\omega \epsilon_0 (n_{co}^1)^2}{k_{co}^1 \beta} \frac{J_0'(k_{co}^1 \rho_{co}^1)}{J_0(k_{co}^1 \rho_{co}^1)} = \frac{g_{TM}^2}{g_{TM}^1}$$

- After we specified the fiber parameters and chose the frequency ω, the propagation constants of the TM modes can be found by solving for β.

$$\mathcal{A}_{TM} = \frac{g_{TM}^1}{J_0(k_{co}^1 \rho_{co}^1) \sqrt{k_{cl}^1 \rho_{cl}^1}} f_{TM}$$

Solutions for the Guided Modes



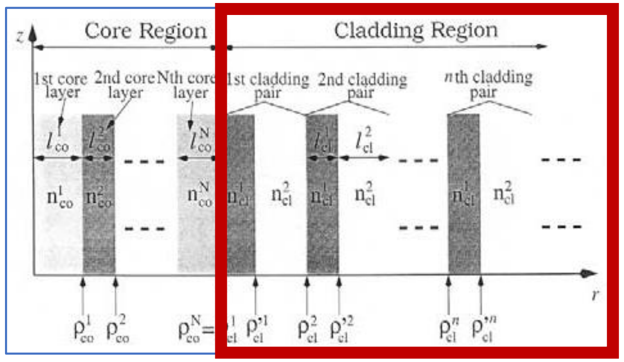
- After we found β, we can go back to the first split equation and find $\mathcal{A}_{TM}$

$$\begin{bmatrix} J_l(k_{co}^1\rho_{co}^1) & 0 \\ \frac{\omega\epsilon_0(n_{co}^1)^2}{k_{co}^1\beta}J_l'(k_{co}^1\rho_{co}^1) & \frac{l}{(k_{co}^1)^2\rho_{co}^1}J_l(k_{co}^1 r) \end{bmatrix} \cdot \begin{bmatrix} \mathcal{A}_{TM} \\ \mathcal{C}_{TE} \end{bmatrix} = \frac{1}{\sqrt{k_{cl}^1\rho_{cl}^1}} \begin{bmatrix} g_{TM}^1 & g_{TE}^1 \\ g_{TM}^2 & g_{TE}^2 \end{bmatrix} \begin{bmatrix} f_{TM} \\ f_{TE} \end{bmatrix}$$

$$\mathcal{A}_{TM} = \frac{g_{TM}^1}{J_0(k_{co}^1\rho_{co}^1)\sqrt{k_{cl}^1\rho_{cl}^1}} f_{TM}$$

Important Note:
This result relates the mode amplitude $\mathcal{A}_{TM}$ in the **first core layer** to f_{TM}, which determines the fields within the **entire fiber cladding region**

Solutions for the Guided Modes



- For TE modes, we can get similar solution using the same method with $\mathcal{A}_{TM} = 0$:
- To find the propagation constant β:

$$\frac{\omega\mu_0}{k_{co}^1\beta}\frac{J_0'(k_{co}^1\rho_{co}^1)}{J_0(k_{co}^1\rho_{co}^1)} = \frac{g_{TE}^4}{g_{TE}^3}$$

- And the relation between $\mathcal{C}_{TE}$ and f_{TE} are:

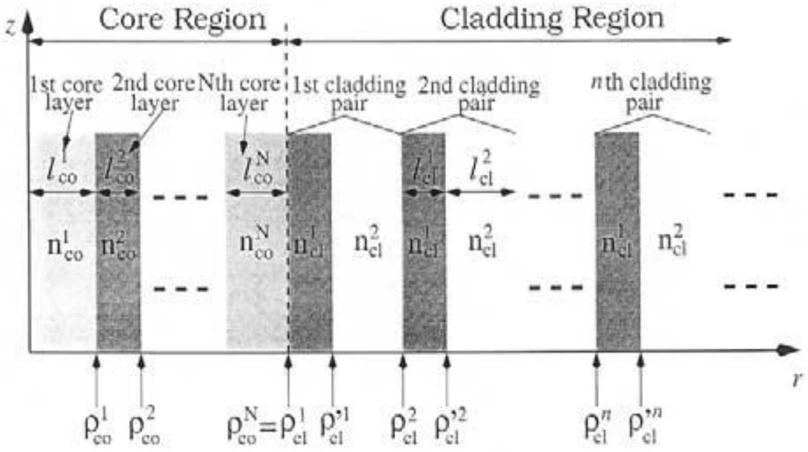
$$\mathcal{C}_{TE} = \frac{g_{TE}^3}{J_0(k_{co}^1\rho_{co}^1)\sqrt{k_{cl}^1\rho_{cl}^1}} f_{TE}$$

- To complete the fields coefficients at the core, we can choose the normalization factor of the guided mode such as $\mathcal{A}_{TM} = 1$ or $\mathcal{C}_{TE} = 1$.
- At this point, we have all the information needed for finding the fields in the core and cladding regions.

Core and Cladding fields solutions

- Now we have a full process to find the variables and the fields of the Bragg fiber.

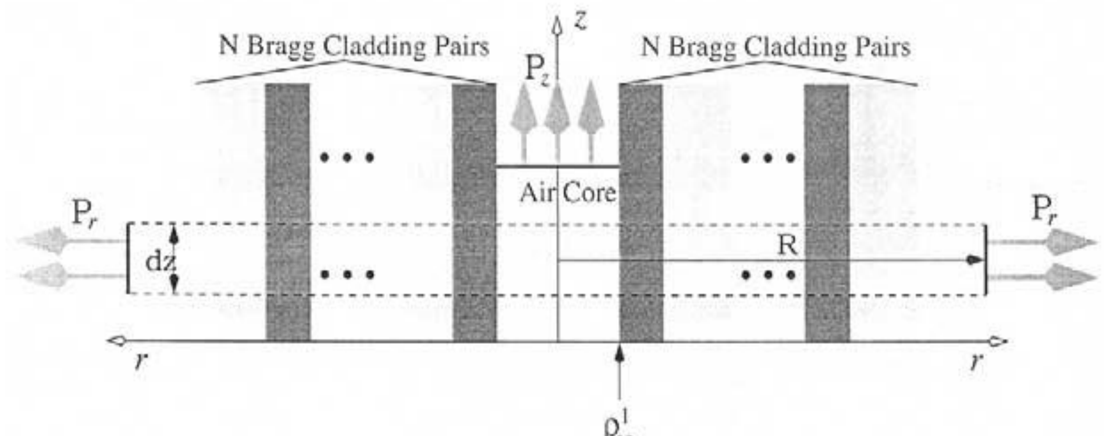
1. **Cladding** — Find the asymptotic approximation coefficients a_n, b_n, c_n, d_n
2. **Cladding** — Calculate the Fields E & H for all the fiber cladding layers
3. **Core** — Define the coefficients of the first core layer ($\mathcal{A}_{TM}$ or $\mathcal{C}_{TE}$)
4. **Core** — Find the transfer matrix T using $M^{-1} \cdot M$
5. **Core** — Applying the propagation of the field coefficients using transfer matrix T
6. **Core** — Calculate the Fields E & H for all the fiber core layers



Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Radiation Loss

- For an air core Bragg fiber, the propagation loss is the sum of 2 terms:
  - Loss through the finite number of cladding
  - Absorption loss due to the cladding materials (Not consider here)
- The radiation loss depends mostly on the **index contrast** of the cladding media and the **number of cladding pairs**.
- We will use the asymptotic theory to estimate the number of cladding pairs needed to reduce the radiation loss below $0.2\ dB/km$
- To simplify the calculation, we using Bragg fiber with one air core layer, bounded by N pairs of cladding layers.



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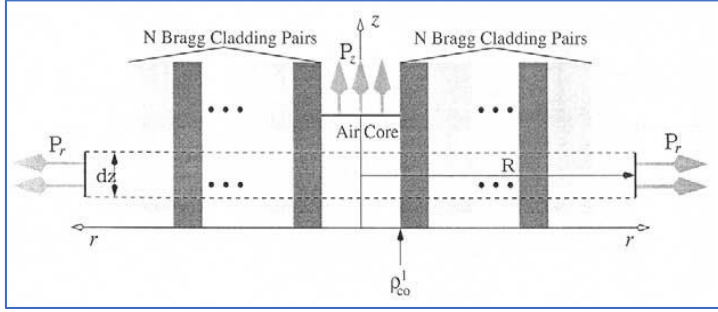
Radiation Loss



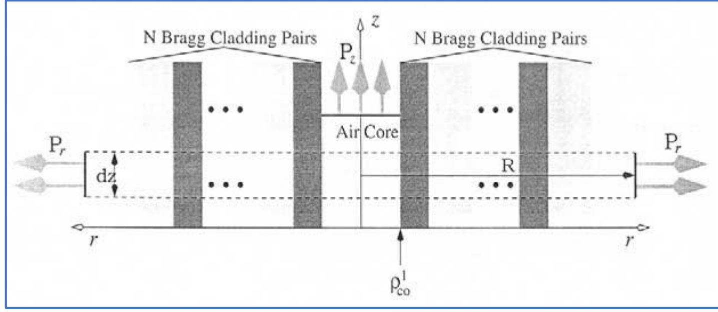
- As we develop before, we want to calculate the fields of both core and cladding.
- Because we have only 1 layer in the core, the transfer matrix T will be an **identity matrix**.
- In turn, the g parameters become:

$$g^3_{TE} = \lambda_{TE} - A_{TE} + B_{TE}$$
$$g^4_{TE} = -\frac{i\omega\mu_0}{k^1_{cl}\beta}(\lambda_{TE} - A_{TE} - B_{TE})$$
$$g^1_{TM} = \lambda_{TM} - A_{TM} + B_{TM}$$
$$g^2_{TM} = -\frac{i\omega\epsilon_0\left(n^1_{cl}\right)^2}{k^1_{cl}\beta}t_{j2}(\lambda_{TM} - A_{TM} - B_{TM})$$

- Considering first the TE mode, the fields in the core are:

$$H_z(r) = \mathcal{C}_{TE}J_0(k^1_{co}r)$$
$$E_\theta = -i\frac{\omega\mu_0}{k^1_{cl}}\mathcal{C}_{TE}J_0'(k^1_{co}r)$$
$$H_r = i\frac{\beta}{k^1_{cl}}\mathcal{C}_{TE}J_0'(k^1_{co}r)$$

Radiation Loss



$$H_z(r) = \mathcal{C}_{TE} J_0(k_{co}^1 r)$$
$$E_\theta = -i \frac{\omega \mu_0}{k_{cl}^1} \mathcal{C}_{TE} J_0'(k_{co}^1 r)$$
$$H_r = i \frac{\beta}{k_{cl}^1} \mathcal{C}_{TE} J_0'(k_{co}^1 r)$$

- From these expressions for E_θ and H_r, we find the **power flux along the direction z in the low index core**:

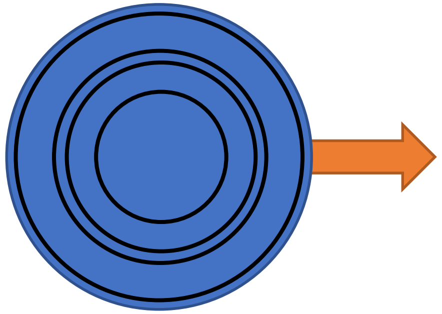
$$P_z^{TE} = |\mathcal{C}_{TE}|^2 \frac{\pi \omega \mu_0 \beta}{(k_{cl}^1)^2} \int_0^{\rho_{co}^1} [J_0'(k_{co}^1 r)]^2 r dr$$

- The fields at the cladding (asymptotic solutions) consist of 2 components: outgoing wave (c_N) and incoming wave (d_N).

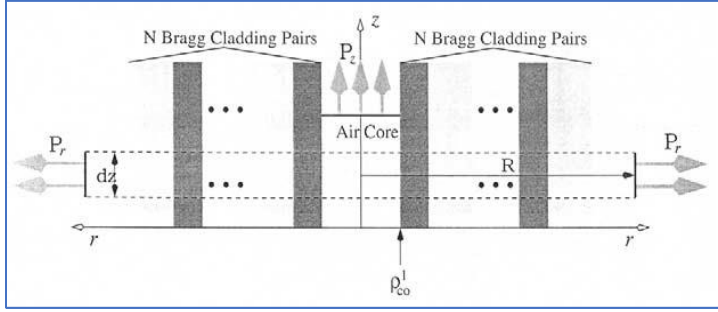
- The radiation field outside N layers Bragg fiber, can be well approximate by calculate only the **outgoing component** of the fields of the $N+1$ layer ($d_{N+1} = 0$). The fields at the $N+1$ cladding layer:

$$H_z = \frac{f_{TE}}{\sqrt{k_{cl}^1 r}} \left(c_{N+1} e^{i k_{cl}^1 (r - \rho_{cl}^{N+1})} + d_{N+1} e^{-i k_{cl}^1 (r - \rho_{cl}^{N+1})} \right)$$

$$E_\theta = \frac{\omega \mu_0}{k_{cl}^1} \frac{f_{TE}}{\sqrt{k_{cl}^1 r}} \left(c_{N+1} e^{i k_{cl}^1 (r - \rho_{cl}^{N+1})} - d_{N+1} e^{-i k_{cl}^1 (r - \rho_{cl}^{N+1})} \right)$$



Radiation Loss



- Using the last equation, and taking $d_{N+1} = 0$: we can calculate the radial flux:

$$H_z = \frac{f_{TE}}{\sqrt{k_{cl}^1 r}} c_{N+1} e^{ik_{cl}^1(r-\rho_{cl}^{N+1})} \quad ; \quad E_\theta = \frac{\omega\mu_0}{k_{cl}^1} \frac{f_{TE}}{\sqrt{k_{cl}^1 r}} c_{N+1} e^{ik_{cl}^1(r-\rho_{cl}^{N+1})}$$

- Hence, the radial flux is:

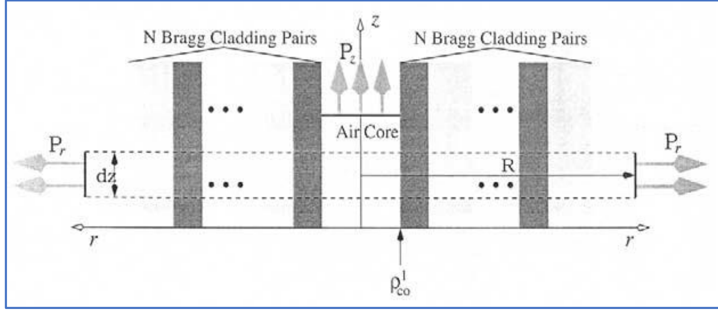
$$P_r^{TE} = \frac{\pi\omega\mu_0}{(k_{cl}^1)^2} f_{TE}^2 (c_{N+1})^2 dz$$

- For TE modes propagating along the direction of the Bragg fiber, with the presence of radiation loss, the optical power decays as $\exp(-\alpha_{TE} z)$, where α_{TE} is the radial loss constant.

- From the definition of P_z^{TE} and P_r^{TE}, we can calculate α_{TE} as:

$$\alpha_{TE} = \frac{P_r^{TE}}{P_z^{TE} dz} = \frac{1}{\beta}\left(\frac{k_{co}^1}{k_{cl}^1}\right)^2 \left|\frac{B_{TE}}{\lambda_{TE} - A_{TE} + B_{TE}}\right|^2 |\lambda_{TE}|^{2N} \cdot \frac{[J_0(k_{co}^1 \rho_{co}^1)]^2 k_{cl}^1 \rho_{cl}^1}{\int_0^{\rho_{co}^1} r dr\, [J_0'(k_{co}^1 r)]^2}$$

Radiation Loss



- For TM mode we can follow the same procedure and get:

$$\alpha_{TM} = \frac{P_r^{TM}}{P_z^{TM} dz} = \frac{1}{\beta}\left(\frac{n_{cl}^1 k_{co}^1}{n_{co}^1 k_{cl}^1}\right)^2 \left|\frac{B_{TM}}{\lambda_{TM} - A_{TM} + B_{TM}}\right|^2 |\lambda_{TM}|^{2N} \cdot \frac{[J_0(k_{co}^1 \rho_{co}^1)]^2 k_{cl}^1 \rho_{cl}^1}{\int_0^{\rho_{co}^1} r dr\, [J_0'(k_{co}^1 r)]^2}$$

- In order to have a meaningful meaning and feeling for the radiation loss (order of magnitude), we will use the following assumption, to simplify our result:

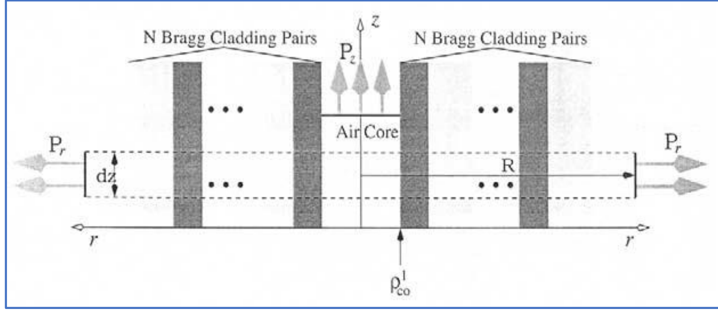
  1. Define new parameter - $x = k_{co}^1 \rho_{co}^1$.
  2. For Bessel function, we can use - $J_0'(x) = -J_1(x)$.
     For order-of-magnitude estimate, we will choose $x = 3.8317$ (first zero point of $J_1(x)$)
  3. For Bessel function: $\int_0^x du\, u[J_1(u)]^2 = x^2[J_2(x)]^2/2$

- The last component in the right part of the equation become:

$$\frac{[J_0(k_{co}^1 \rho_{co}^1)]^2 k_{cl}^1 \rho_{cl}^1}{\int_0^{\rho_{co}^1} dr\, r[J_0'(k_{co}^1 r)]^2} = k_{co}^1 k_{cl}^1 \left.\frac{[J_0(x)]^2}{\int_0^x du\, u\, [J_1(u)]^2}\right|_{x=3.8317} \approx 0.522 k_{co}^1 k_{cl}^1$$

Mathematical Methods of Physics by Jon Mathews, Robert L. Walker

Radiation Loss



- From the definition of $A_{TE}, A_{TM}, B_{TE}, B_{TM}, \lambda_{TE}, \lambda_{TM}$ we can see that they have the same order of magnitude. Therefore, we can take the following arguments to be equal 1:

$$\frac{B_{TE}}{\lambda_{TE} - A_{TE} + B_{TE}} \approx 1 \quad ; \quad \frac{B_{TM}}{\lambda_{TM} - A_{TM} + B_{TM}} \approx 1$$

- Combining this approximations, we can write:

$$\alpha_{TE} = 0.522 \frac{(k_{co}^1)^3}{\beta k_{cl}^1} |\lambda_{TE}|^{2N}$$

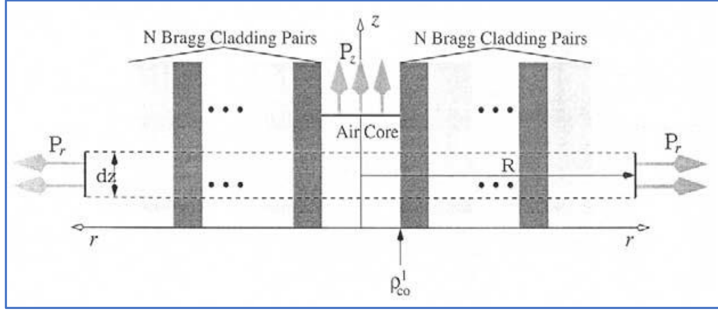
$$\alpha_{TM} = 0.522 \left(\frac{n_{cl}^1}{n_{co}^1}\right)^2 \frac{(k_{co}^1)^3}{\beta k_{cl}^1} |\lambda_{TM}|^{2N}$$

- If we take some number in:

$$n_{co}^1 = 1\ (air)\ ;\ \lambda = 2\pi c/\omega = 1.55\mu m$$

$$\boldsymbol{Assuming}:\ \beta = k_{co}^1 = \omega/\sqrt{2}c \quad ; \quad k_{cl}^1 = n_{cl}^1 \omega/c$$

Radiation Loss



- The radiation loss constants for TE and TM mode (in dB/km):

$$\alpha_{TE}(dB/km) = 4.6 \cdot 10^9 \frac{1}{n_{cl}^1} |\lambda_{TE}|^{2N}$$

$$\alpha_{TM}(dB/km) = 4.6 \cdot 10^9 \cdot n_{cl}^1 |\lambda_{TM}|^{2N}$$

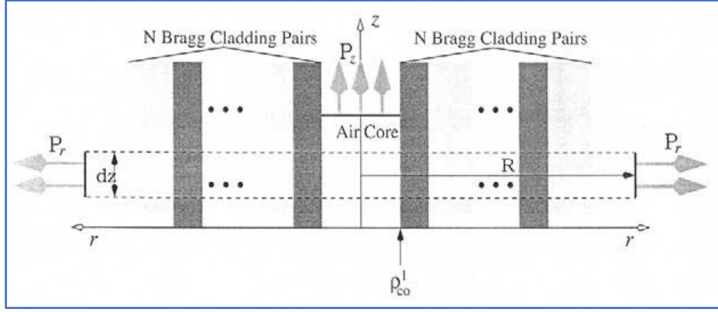
- The values of λ_{TE} and λ_{TM} have complicated dependence on $\beta, n_{cl}^1, l_{cl}^1, n_{cl}^2$ and l_{cl}^2.

- However, when the cladding layers form quarter wave stack (meaning $k_{cl}^1 l_{cl}^1 = k_{cl}^2 l_{cl}^2 = \pi/2$), such that light is optimally confined, the expressions for $|\lambda_{TE}|$ and $|\lambda_{TM}|$ take simpler forms:

$$|\lambda_{TE}| = \min\left(\frac{k_{cl}^2}{k_{cl}^1}, \frac{k_{cl}^1}{k_{cl}^2}\right)$$

$$|\lambda_{TM}| = \min\left[\left(\frac{n_{cl}^2}{n_{cl}^1}\right)^2 \frac{k_{cl}^1}{k_{cl}^2}, \left(\frac{n_{cl}^1}{n_{cl}^2}\right)^2 \frac{k_{cl}^2}{k_{cl}^1}\right]$$

Results – Number of Layers



- Using the arguments until now, we can calculate the number of layers needed in the cladding to achieve loss equal or less than $0.2dB/km$.

- We choose cladding layer 2, to be the low index medium with $n_{cl}^2 = 1.5$ (silica glass for example)

- For this index, for $0 < \beta < \omega/c$, the minimum value of λ are:

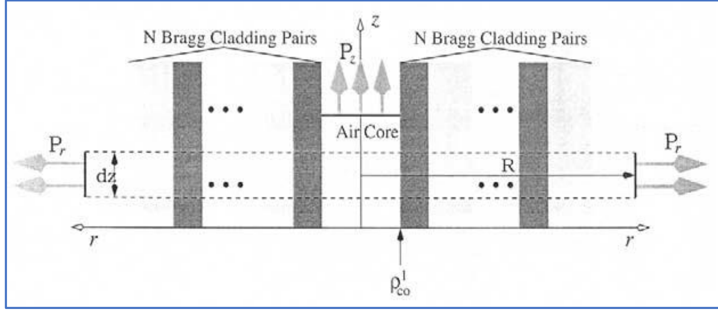
$$|\lambda_{TE}| = \sqrt{[(n_{cl}^2)^2-1]/[(n_{cl}^1)^2-1]} \quad ; \quad |\lambda_{TM}| = n_{cl}^2/n_{cl}^1$$

- Substituting this value into $\alpha_{TE/TM}$ we find that the minimum number of Bragg layers pairs required to achieve $0.2dB/km$ is:

$$N_{TE} = \frac{23.9 - \ln(n_{cl}^1)}{\ln[(n_{cl}^1)^2 - 1] - \ln[(n_{cl}^2)^2 - 1]}$$

$$N_{TM} = \frac{23.9 + \ln(n_{cl}^1)}{2[\ln(n_{cl}^1) - \ln(n_{cl}^2)]}$$

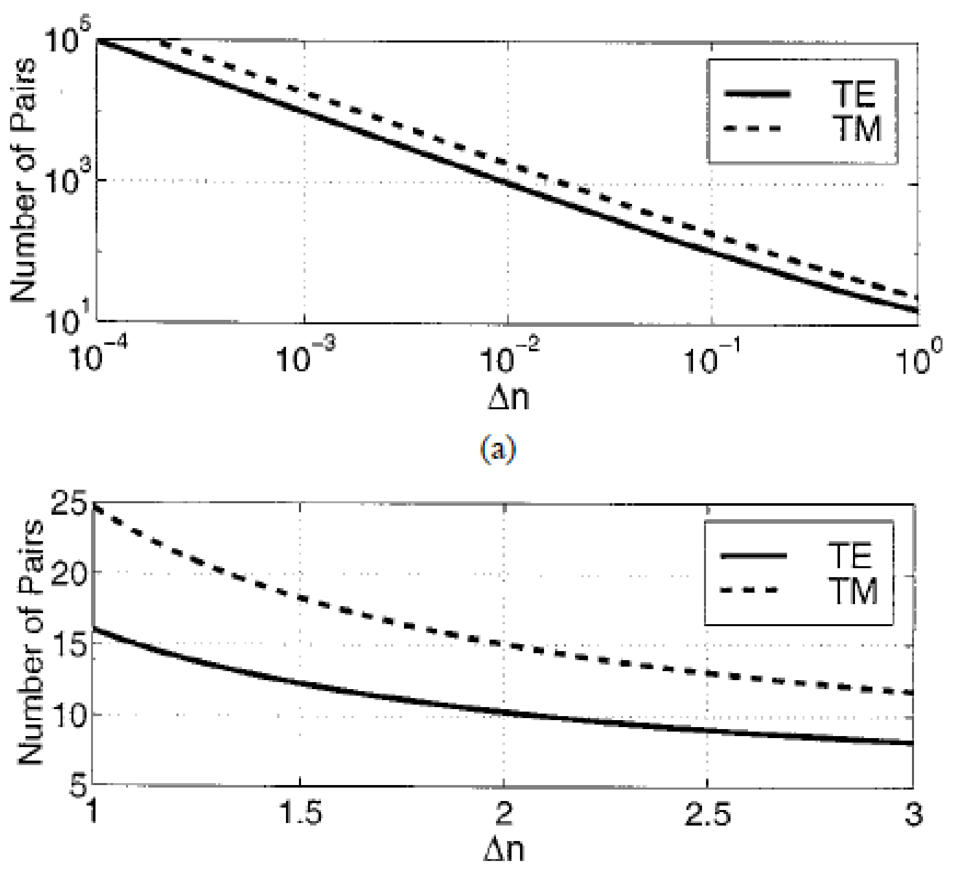
Results – Number of Layers



- $\Delta n = n_{cl}^1 - n_{cl}^2$ - index contrast
  - $\Delta n < 0.01 \quad \rightarrow \quad N > 1000$
  - $0.1 < \Delta n < 1 \quad \rightarrow \quad N < 200$
  - $1 < \Delta n < 3 \quad \rightarrow \quad 12 < N < 25$

For less than $0.2dB/km$ for both TE & TM

- In the **asymptotic limit**, the mixed modes ($l \neq 0$) in the cladding structure can always be classified into TE and TM components.
- Therefore, their radiation loss is determined by the TM component, because TM component is less confined and suffers more radiation loss compared with TE component, as can be seen from the figure.
- ***NOTICE***: We used the smallest possible values of $\lambda_{TE/TM}$, corresponding to min. number of Bragg pairs for $0.2dB/km$ loss
- Better estimation requires values of $\beta, n_{cl}^1, l_{cl}^1, n_{cl}^2, l_{cl}^2$ for exact calculation of $|\lambda_{TE/TM}|$ and $\alpha_{TE/TM}$.

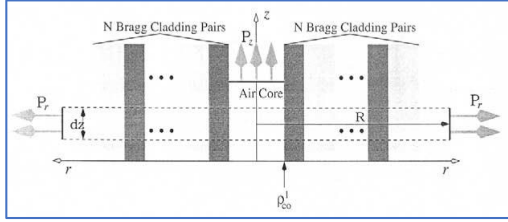


(a)

(b)

Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Results – Comparison to FDTD



- We will compare our method to a well-known method called FDTD (Finite Difference Time-Domain)

- We will compare the dispersion properties of the Bragg fiber – with 2 graphs:
  - $n_{eff}(=\beta c/\omega)\ \ vs\ \ \omega[2\pi c/\Lambda]$
  - $\omega[2\pi c/\Lambda]\ vs\ \ \beta[2\pi/\Lambda]$

- The Bragg fiber will be comprised from:
  - **Cladding**: $n_{cl}^{1} = 4.6\ , l_{cl}^{1} = 0.25\Lambda\ \ ,\ \ n_{cl}^{2} = 1.5\ , l_{cl}^{2} = 0.75\Lambda\ \ \backslash\backslash\ \ \text{where}\ \Lambda = l_{cl}^{1} + l_{cl}^{2}$
  - **Core**: cosist from 5 layers: $n_{co}^{1} = 1\ \ ,\ \ \rho_{co}^{1} = 1\Lambda$

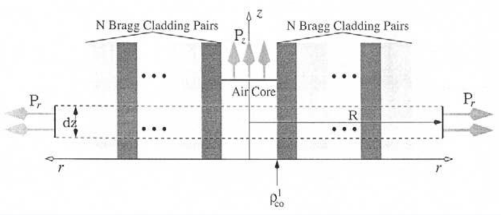
$$n_{co}^{2} = n_{co}^{4} = 4.6\ \ ,\ \ n_{co}^{3} = n_{co}^{5} = 1.5\ \ ;\ \ l_{co}^{2} = l_{co}^{4} = 0.25\Lambda\ \ , l_{co}^{3} = l_{co}^{5} = 0.75\Lambda$$

- For the FDTD method we will choose $\Lambda = 24$ (number of cells for calculation).

- For index contrast we have chosen, 10 cladding pairs are enough to reduce the radiation loss to approximately 0.2 dB/km (as shown before). **We will use 3 cladding pairs** (+ 5 core layers).

Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Results – Comparison to FDTD

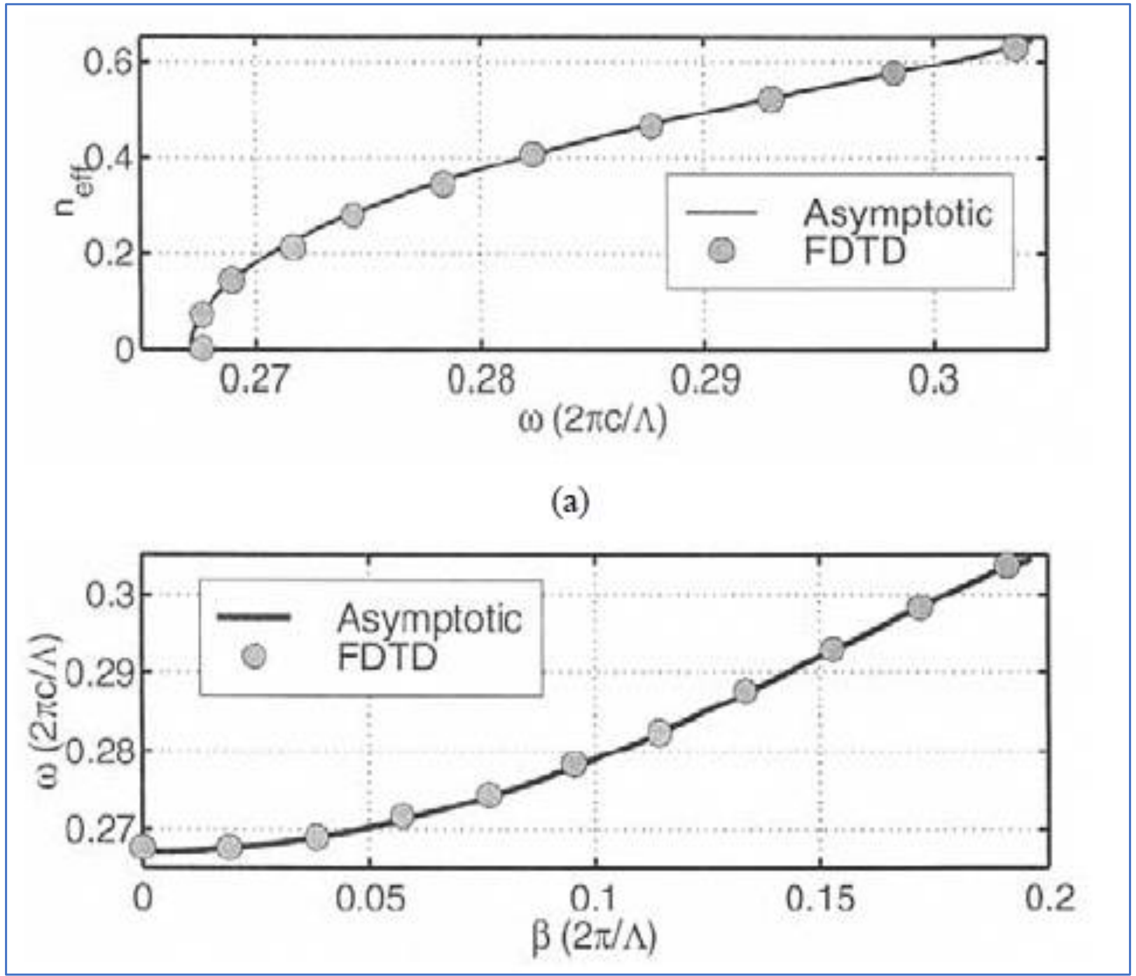


- Both the asymptotic analysis and FDTD calculation show that the Bragg fiber **supports a guided mode** propagating in the air core.

- The results shown for $l = 0$, meaning the azimuthal dependence of the mode is $\cos(\theta)$ or $\sin(\theta)$

- The method agreed well with each other for the dispersion behavior.

- The main source for the small difference is the numerical error in the finite difference time domain algorithm, which can be improve using more calculation cells.



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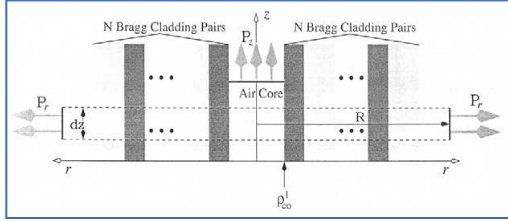
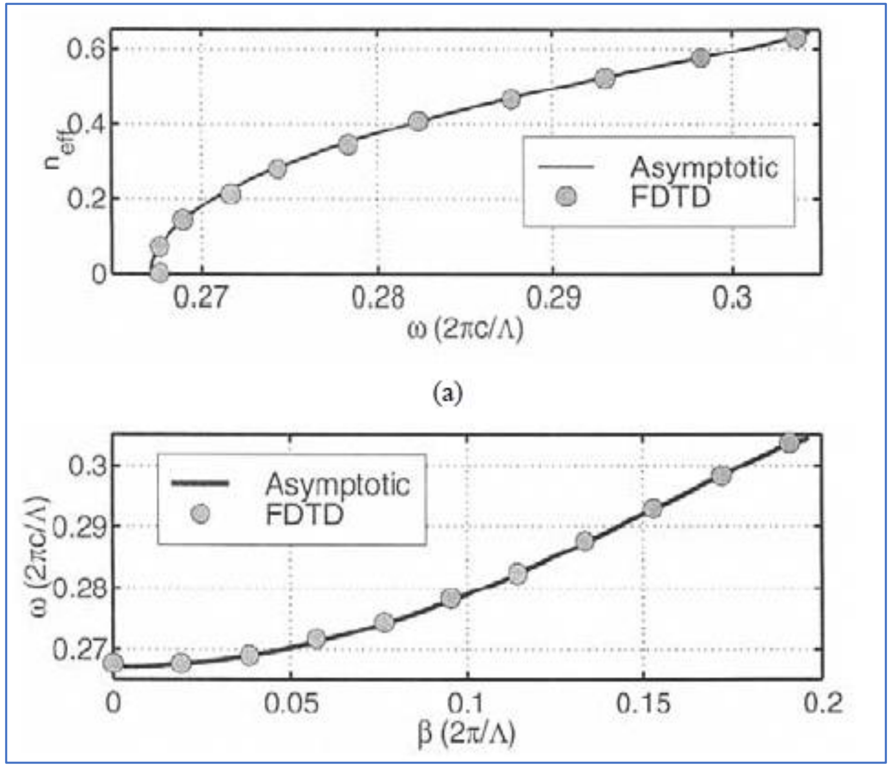
Results – Comparison to FDTD



- ***How is n_{eff} is smaller than 1?***
  - β is the imaginary part of the propagation constant, which effect on the phase of the wave.
  - The relation between them is:

$$\beta = n_{eff}\frac{2\pi}{\lambda}$$

  - In conventional fibers, because we leaning on total internal reflection (TIR) mechanism, the value of n_{eff} will always lay in:
    $$n_{cladding} < n_{eff} < n_{core}$$
    because β must comply the condition of the critical angle
  - In our method, where the TIR is no longer an issue, we can get lower value of β



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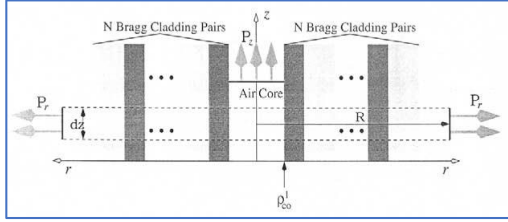
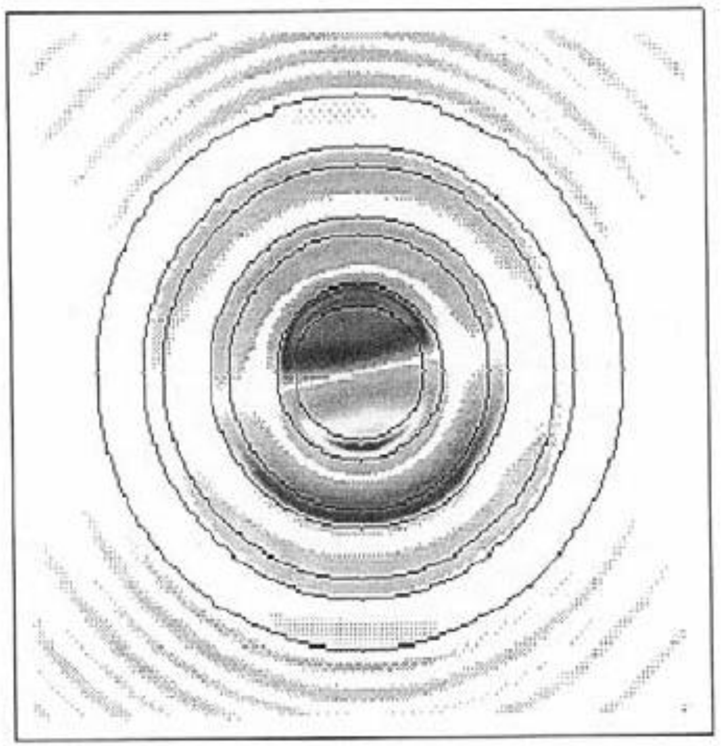
Results – Comparison to FDTD

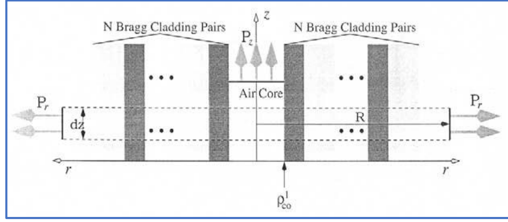


- The distribution of the field H_z obtained from the calculation, is shown in the figure.
- The frequency and propagation constant of the mode are $\omega = 0.291(2\pi c/\Lambda)$ and $\beta = 0.143(2\pi/\Lambda)$ respectively.
- We can see that a guided mode has an azimuthal number of $l = 1$.
- Most of the field contained within the air core and the first cladding layer.
- Because we used a small number of layers, a radiation field outside of the Bragg fiber has been developed.



Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Results – Number of Core Layers

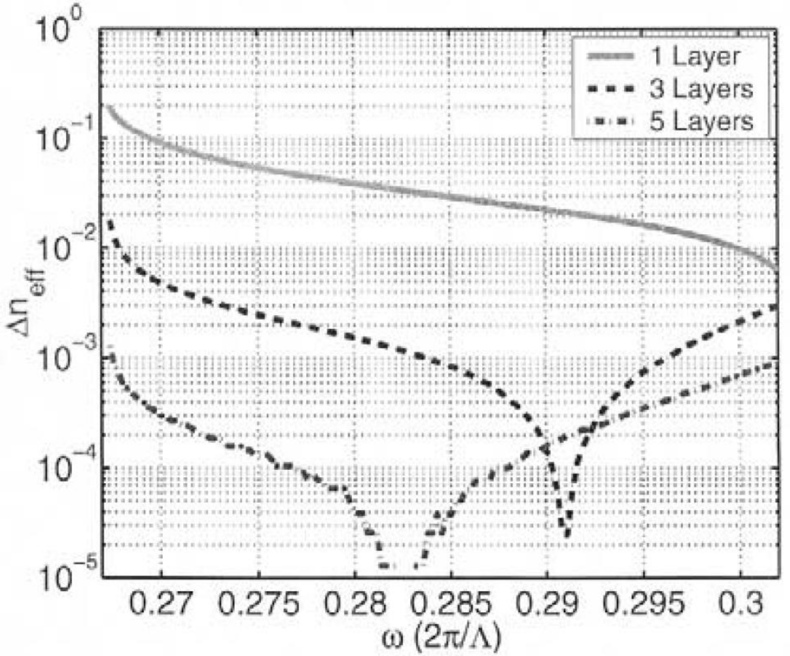


- A main advantage of this asymptotic analysis is that the result can be arbitrarily precise by incorporating more and more layers into the core.

- The asymptotic results obtained using an inner core region consist of N dielectric layers, should converge as a function of N to the exact solution.

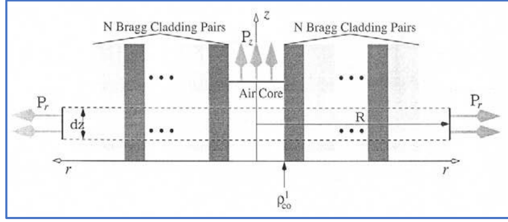
- To show that behavior, we choose a core with 7 layers and calculate its effectivce index n_{eff}^{7} as base.
  Then we compare the result to core with 1/3/5 layers. Where Δn_{eff} defined as $\left|n_{eff}^{i} - n_{eff}^{7}\right|$.

- Results:
  - At 1 layer, the difference go up to 0.2, which is quite significant.
  - Addition of one more pair (3 layers combined), reduce the difference to 0.02
  - With 5 layers, it go down to 0.001 .



Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.

Results – Field Distribution comparison



- The field itself can be calculated using the steps we introduced:

1. **Cladding** – Find the asymptotic approximation coefficients
2. **Cladding** – Calculate the Fields E & H for all the fiber cladding layers
3. **Core** – Define the coefficients of the first core layer ($\mathcal{A}_{TM}$ or $\mathcal{C}_{TE}$)
4. **Core** – Find the transfer matrix T using $M^{-1} \cdot M$
5. **Core** – Applying the propagation of the field coefficients using transfer matrix T
6. **Core** – Calculate the Fields E & H for all the fiber core layers

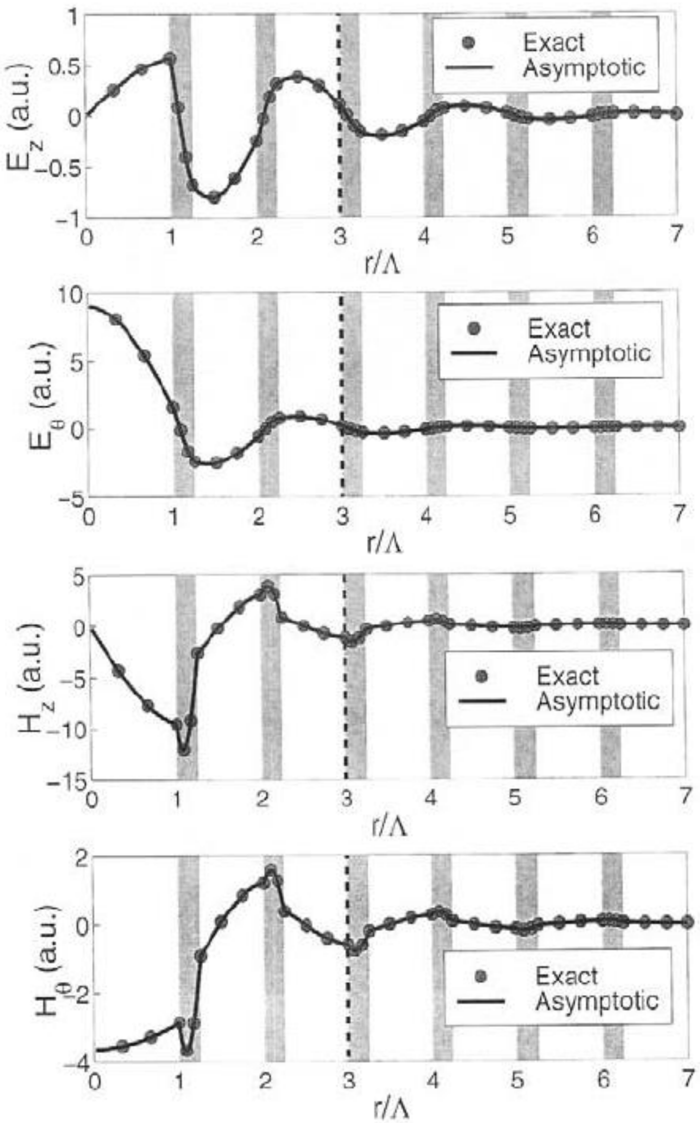
- We apply this algorithm to study the field distribution of the guided Bragg fiber mode at $\omega = 0.286(2\pi c/\Lambda)$, using a core region of five layers, we find the propagation constant to be $\beta = 0.128(2\pi/\Lambda)$.

- We compare this calculation to the *"exact solution"* – meaning calculation using the exact calculation of the core to all the cladding layers.

?

Results – Field Distribution comparison

- As anticipated, within the core region, the exact solution and the asymptotic solution are the same (same calculation).
- The accuracy of the approximation is relevant only at the cladding area, and as we can see, is **very small** for all the fields.
- Most of the field is contained in the core and converge to 0 at the cladding layers.
- ***NOTICE***: The free-space wavelength of the mode is $\lambda = 3.5\Lambda$, and the core radius equal to Λ . Their ratio is 0.286, which demonstrate great results with small air core radius.



Xu, Y., Ouyang, G., Lee, R., & Yariv, A. (2002). Asymptotic matrix theory of Bragg fibers. Journal Of Lightwave Technology, 20(3), 428-440.


thank you

Sources Used in This Work

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