Polarization Rotation in Twisted Waveguides and Helical Fibers

Fiber Optics Course by Dr. Alina Karabchevsky

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Outline

1. Polarization Rotation In Curved Fibers

2. Integrated Polarization Rotators

3. Twisted Waveguides

4. Description of Light Propagation in a Twisted Waveguide

Why Do We Need Polarization Rotation Devices?

- Recent developments in integrated silicon photonic devices have demonstrated the remarkable capabilities of the silicon-on-insulator (SOI) platform to implement a wide range of applications, such as optical interconnects, nonlinear photonics, biological sensors, and microspectrometers.
- However, silicon waveguide has large structural birefringence that causes polarization-mode dispersion, polarization-dependent loss, and polarization-dependent wavelength characteristics.
- The polarization-mode dispersion affects devices application in high data rates. The polarization-dependent characteristics limit the application of silicon photonics devices.
- As a result, most of the photonic devices in SOI have been designed for a single mode and a single polarization, typically TE.
- To mimic polarization independent operation, polarization diversity schemes are typically used: the light with arbitrary polarization from the source is split into orthogonal components by a polarization splitter. By further rotating one of the components, a single polarization is achieved.
- A polarization rotator is a key component for such polarization diversity.

[1] J. Zhang, *et al.*, Silicon-Waveguide-Based Mode Evolution Polarization Rotator, IEEE J. Select. Topics Quantum Electron. **16**, 53 (2010)

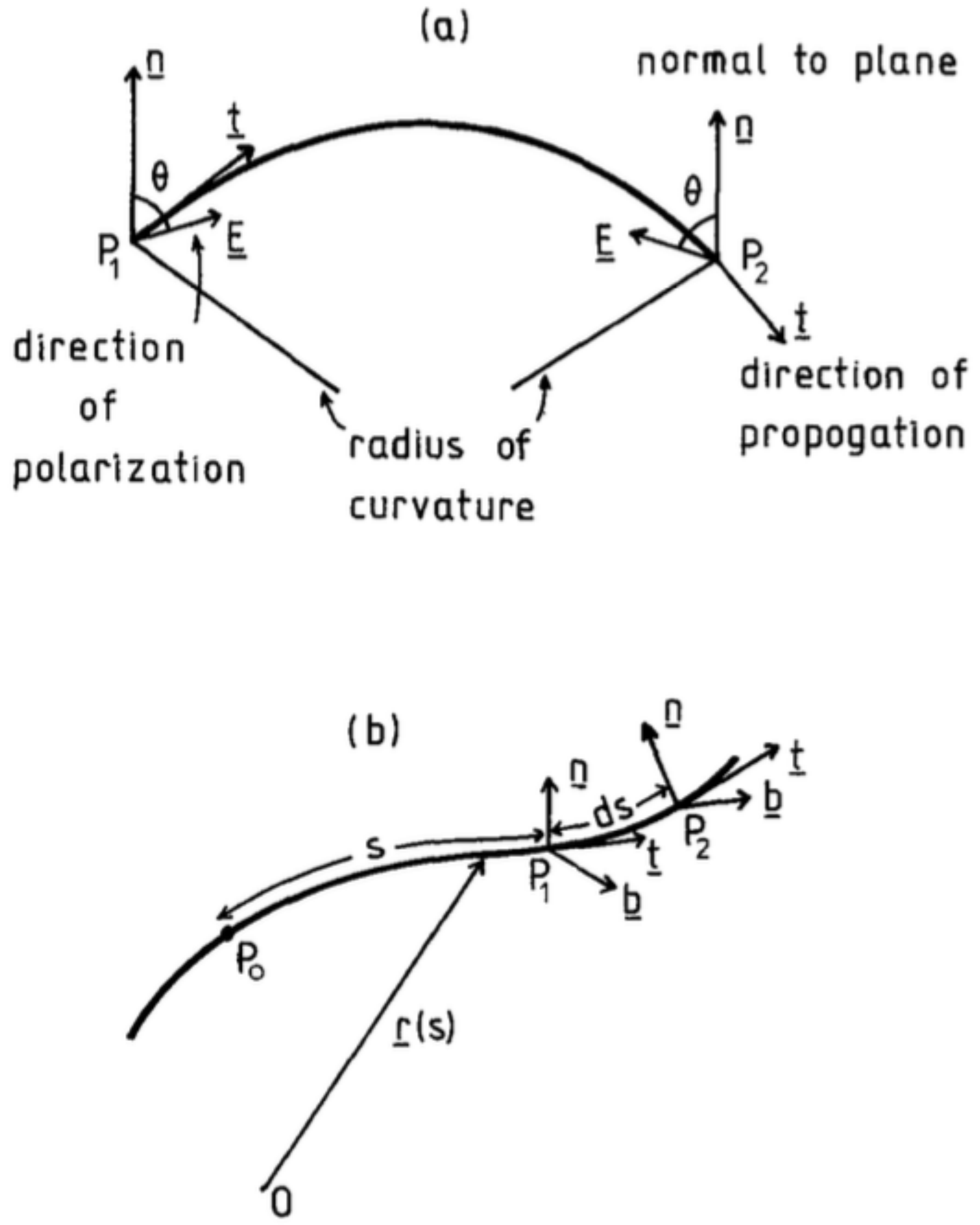
Polarization Rotation in a Curved Fiber

Polarization Rotation in Fiber due to Geometric Effects

- Consider a single mode optical fiber which has no intrinsic linear or circular birefringence.
- Assume that the fiber is bent into a curve lying in a plane and that linearly polarized light is launched into the fiber at a point P_1 and emerges at P_2.
- The direction of propagation at P_1 is not in general parallel to that at P_2, but the direction of the polarization at P_1 and P_2 can both be referred to the normal to the plane (Figure a).

Axiom (supported by experiment):

- Provided that there is no linear or circular birefringence then the angle the polarization makes with the normal is expected to be the same at P_1 and P_2.



Axes used to describe the polarization state: (a) fiber bent in planar curve, (b) fiber bent in general curve.

[1] J. N. Ross, The Rotation of the Polarization in Low Birefringence Monomode Optical Fibres Due to Geometric Effects, Opt. Quant. Electron. **16**, 455 (1984)

Polarization Rotation in Fiber due to Geometric Effects

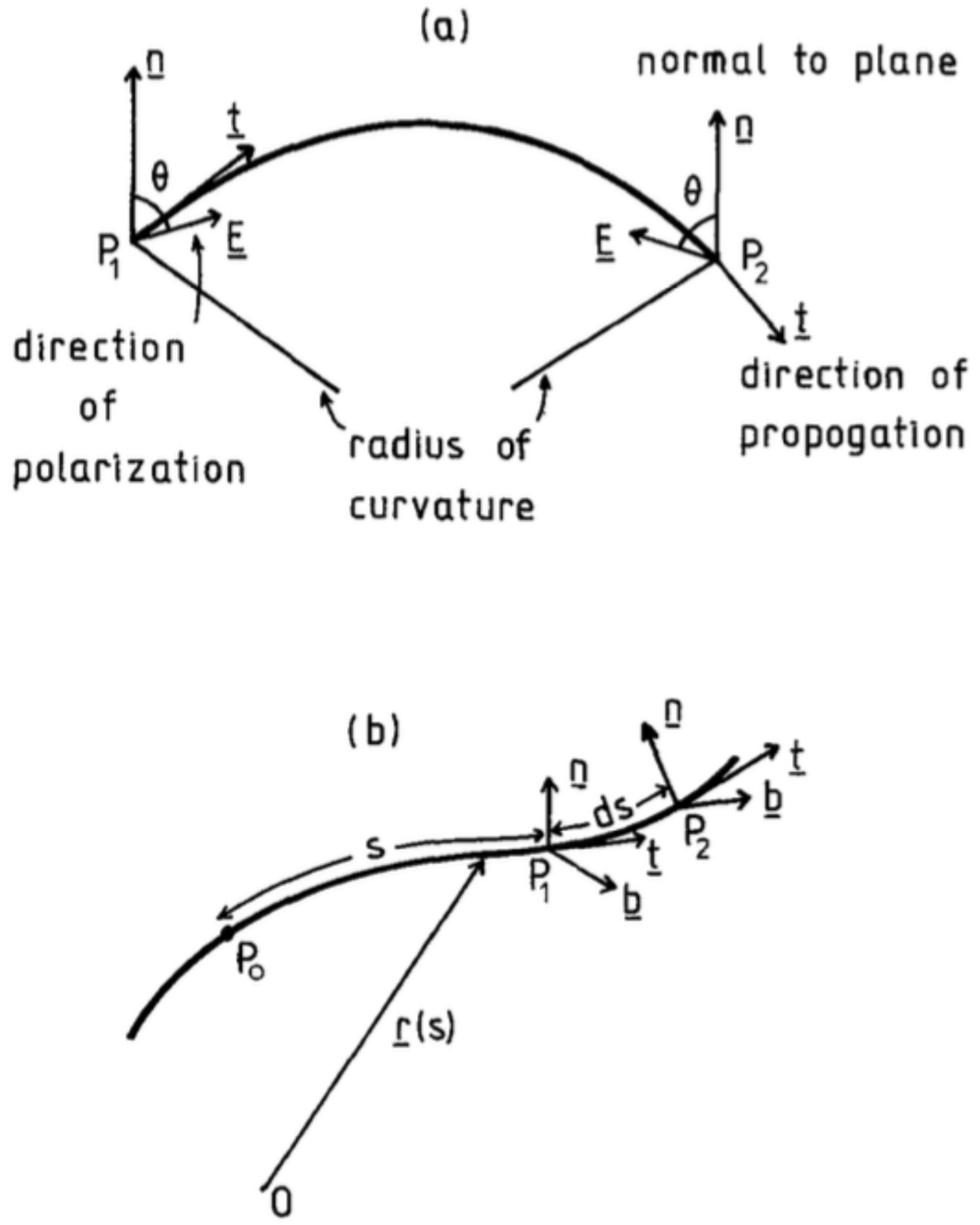
- Now Consider the fiber with its axis lying along a curve $\mathbf{r}(s)$ where s is the distance along the curve from an arbitrary point P_0.

- At point P_1 a triad of orthogonal unit vectors $\mathbf{t}_1$, $\mathbf{n}_1$, $\mathbf{b}_1$ may be defined, where $\mathbf{t}_1$ is tangential to the curve, $\mathbf{n_1}$, the *normal* vector, points towards the center of curvature, and $\mathbf{b}_1$ is normal to osculating plane which passes through the center of curvature and is tangential to the curve at P_1

- At a second point P_2 a short distance ds along the curve from P_1 a second orthonormal triad of vectors $\mathbf{t}_2, \mathbf{n}_2, \mathbf{b}_2$ may be defined.

- Provided ds is sufficiently small the angle between $\mathbf{b}_1$and $\mathbf{b}_2$ is τds where τ, the *torsion*, is defined by

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}.$$



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Polarization Rotation in Fiber due to Geometric Effects

- For linearly polarized light traveling along the fiber it is convenient to use the orthonormal set of vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ to define the orientation of the polarization.

- Light is traveling in the direction of $\mathbf{t}$ and in a way analogous to the case of the planar curve the orientation of the polarization is referred to the normal to the osculating plane, i.e. the binormal vector $\mathbf{b}$.

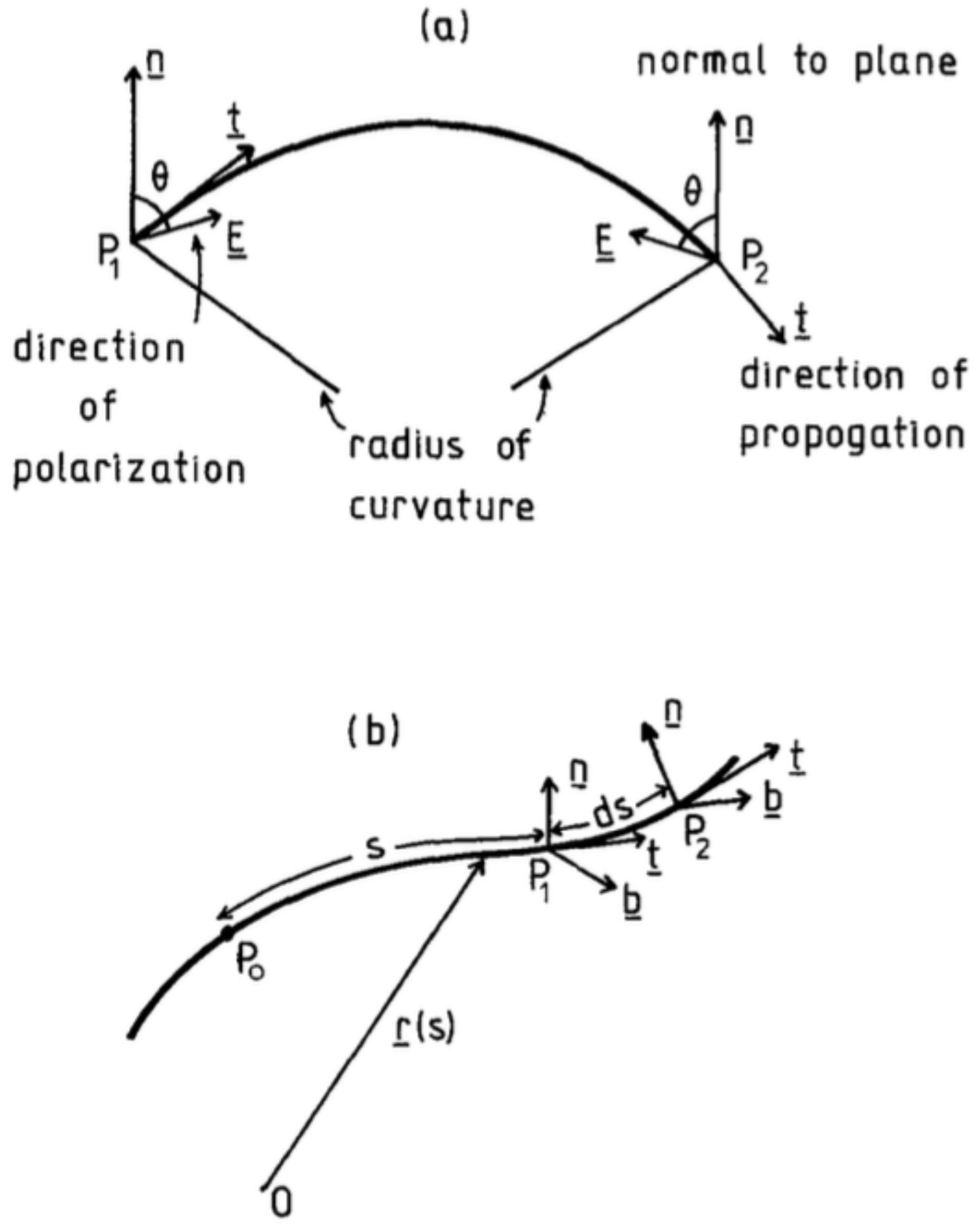
- If the electric field vector makes an angle θ_1, with respect to $\mathbf{b}_1$ at P_1 then it makes the same angle with respect to $\mathbf{b}_1$ at P_2 (provided ds is sufficiently small) in accordance with the axiom stated above. However, with respect to the binormal $\mathbf{b}_2$ at P_2 the angle is

$$\theta_2 = \theta_1 - \tau ds$$

- Thus in general the orientation of the polarization θ_p at a point P with respect to the local binormal vector is related to the polarization orientation θ_0 at P_0 by

$$\theta_p = \theta_0 - \int_{P_0}^{P} \tau ds.$$

- If the curve is planar $\tau = 0$ and $\theta_p = \theta_0$ as required.



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Polarization Rotation in Fiber due to Geometric Effects

- If the optical fiber has a diameter which is not vanishingly small compared with the radius of the bends then it is necessary to include the effect of bend induced linear retardation. The linear retardation introduced by a curvature of κ of the fiber is given by

$$\beta = k_{\mathbf{b}} - k_{\mathbf{n}} = \alpha\kappa^2\rho^2,$$

where $k_{\mathbf{n}}$ and $k_{\mathbf{b}}$ are the wavevectors for light polarized with the electric fields in the $\mathbf{n}$ and $\mathbf{b}$ directions, respectively, ρ is the radius of the optical fiber and α is a constant depending on the wavelength and the elastic properties of the fiber.

- The combination of linear and circular retardation may be analyzed using the Jones N matrix rotation.

- The N matrix describes the way in which the electric field develops as lidght travels through an elemental section of the optical system (in this case the element ds of the fiber). If the electric vector is $\mathbf{E}(s)$ at point $\mathbf{r}(s)$ in the fiber and the N matrix at $\mathbf{r}(s)$ is $N(s)$ then

$$\frac{d\mathbf{E}(s)}{ds} = N(s)\mathbf{E}(s).$$

- The matrix N is found by adding together the matrices for the specific circular and linear birefringence to give, in the local coordinate system

$$N = \begin{pmatrix} (\frac{1}{2}\beta - k)\mathrm{i} & \tau \\ \tau & -(\frac{1}{2}\beta - k)\mathrm{i} \end{pmatrix}.$$

- It is convenient to remove the time dependence and absolute phase of $\mathbf{E}$ by writing

$$\mathbf{E}(s) = \mathbf{A}(s)\mathrm{e}^{\mathrm{i}(\omega t - ks)}.$$

- In this case the equation becomes

$$\frac{d\mathbf{A}(s)}{ds} = N'(s)\mathbf{A}(s),$$

where

$$N' = \begin{pmatrix} \frac{1}{2}\beta\mathrm{i} & \tau \\ -\tau & -\frac{1}{2}\beta i \end{pmatrix}.$$

[1] J. N. Ross, The Rotation of the Polarization in Low Birefringence Monomode Optical Fibres Due to Geometric Effects, Opt. Quant. Electron. **16**, 455 (1984)

Polarization Rotation in a Helical Fiber

- For a fiber wound into a helix an analytic solution for the polarization state is possible as both torsion τ and curvature κ are **constants**. For a helix of radius a and pitch $2\pi b$ the torsion and curvature are given by

$$\tau = \frac{b}{a^2 + b^2},$$
$$\kappa = \frac{a}{a^2 + b^2}.$$

- With constant coefficients the matrix differential equation from the previous slide has the solutions in the form:

$$A_{\mathbf{n}} = \gamma_{\mathbf{n}} \cos \Gamma s + \delta_{\mathbf{n}} \sin \Gamma s$$
$$A_{\mathbf{b}} = \gamma_{\mathbf{b}} \cos \Gamma s + \delta_{\mathbf{b}} \sin \Gamma s'$$

where $\Gamma^2 = \tau^2 + \frac{1}{3}\beta^2$.

- At the start of the fiber $\mathbf{A} = \mathbf{A}_0 = \{A_1, A_2\}$.

- It is readily shown that the electric field at the points at distance s along the fiber is given by

$$\mathbf{A} = M\mathbf{A}_0,$$

where

$$M = S(\theta)GS(\theta)$$

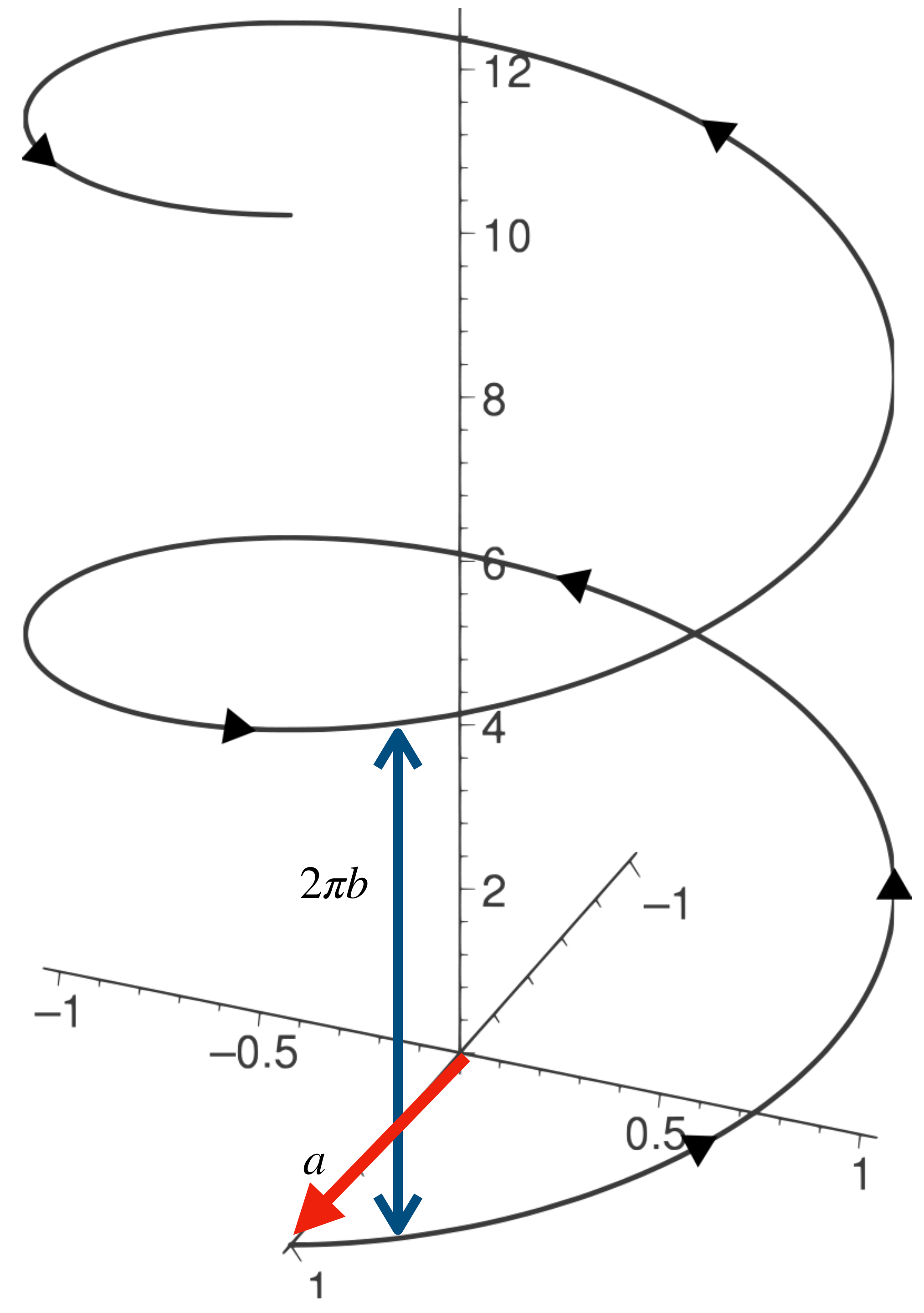
$S(\theta)$ is the rotation matrix

$$S(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

while G is the linear retarder

$$G = \begin{pmatrix} \mathrm{e}^{\mathrm{i}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\gamma} \end{pmatrix}.$$

$$\tan\gamma = \beta(\tan\Gamma s)\cos 2\theta/\Gamma,$$
$$\tan 2\theta = -\tau \tan\Gamma s/\Gamma.$$



Helix with radius a and pitch $2\pi b$.

[1] J. N. Ross, The Rotation of the Polarization in Low Birefringence Monomode Optical Fibres Due to Geometric Effects, Opt. Quant. Electron. **16**, 455 (1984)

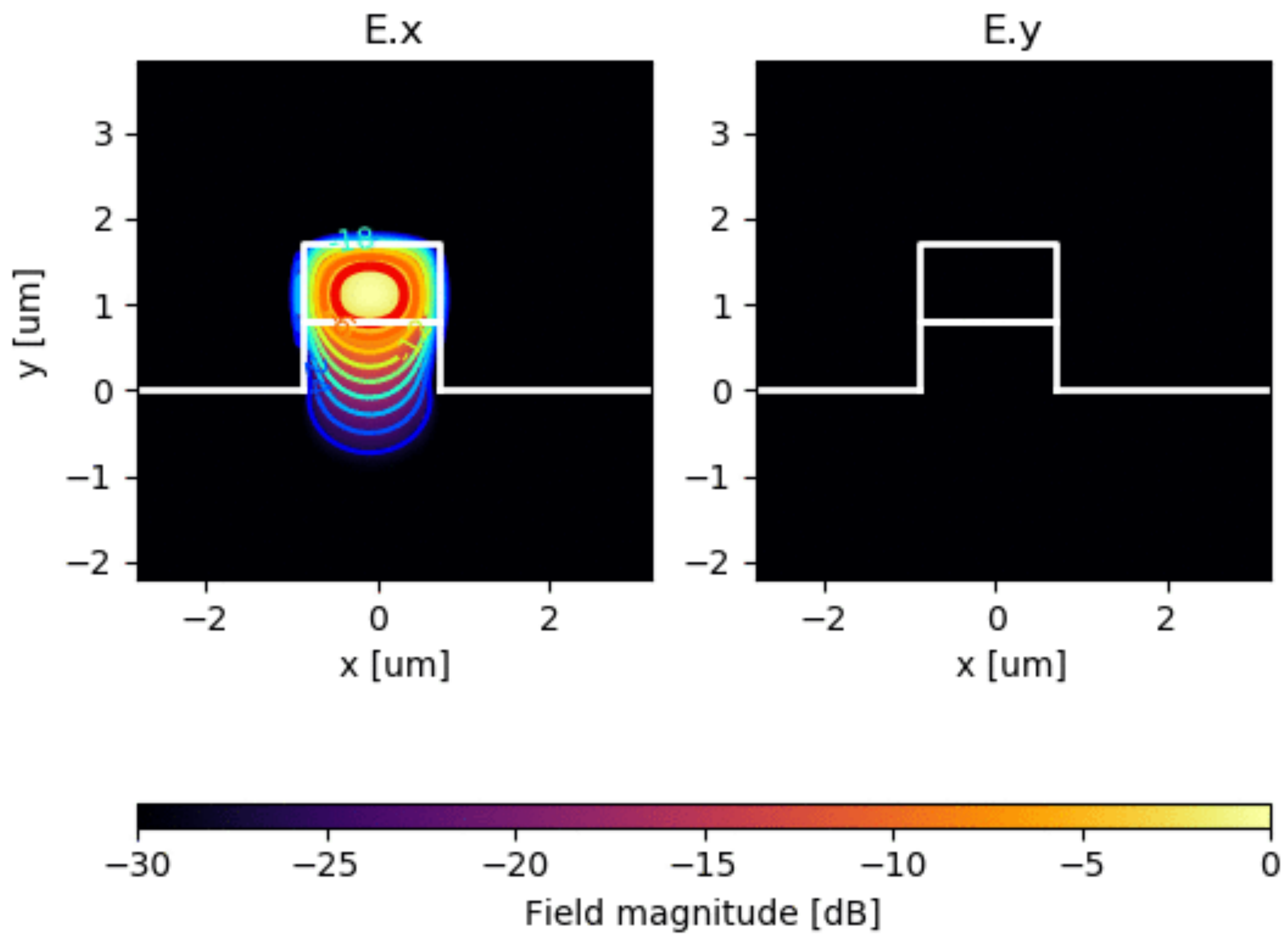
Integrated Polarization Rotators

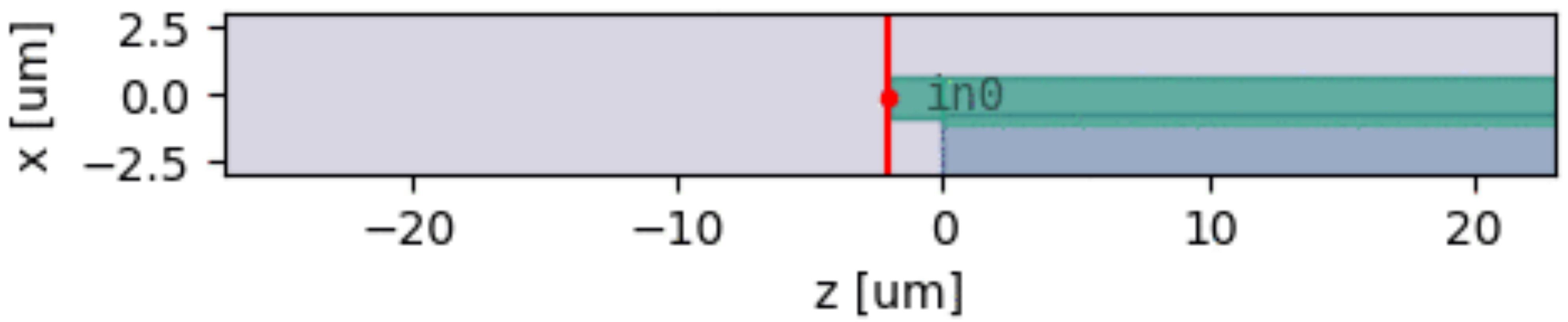
Polarization Rotation in Asymmetric Waveguides

- Waveguides with a symmetric cross section generally support modes that are polarized parallel or perpendicular to the plane of symmetry. For typical integrated planar waveguides, this means that they are polarized parallel and perpendicular to the substrate.

- If a waveguide has no distinct symmetry, it may still support two orthogonal modes, i.e. their angle relative to each other is still 90°. Their absolute angles, however, can take on any value and depend on the waveguide geometry.

- When a horizontally or vertically polarized mode is facet-coupled to such a waveguide, it excites both orthogonal modes. Their interference causes rotation of the field polarization.





[1] M. F. Baier, Polarization Multiplexed Photonic Integrated Circuits for 100 Gbit/s and Beyond, (2018)

Polarization Rotation in Asymmetric Waveguides

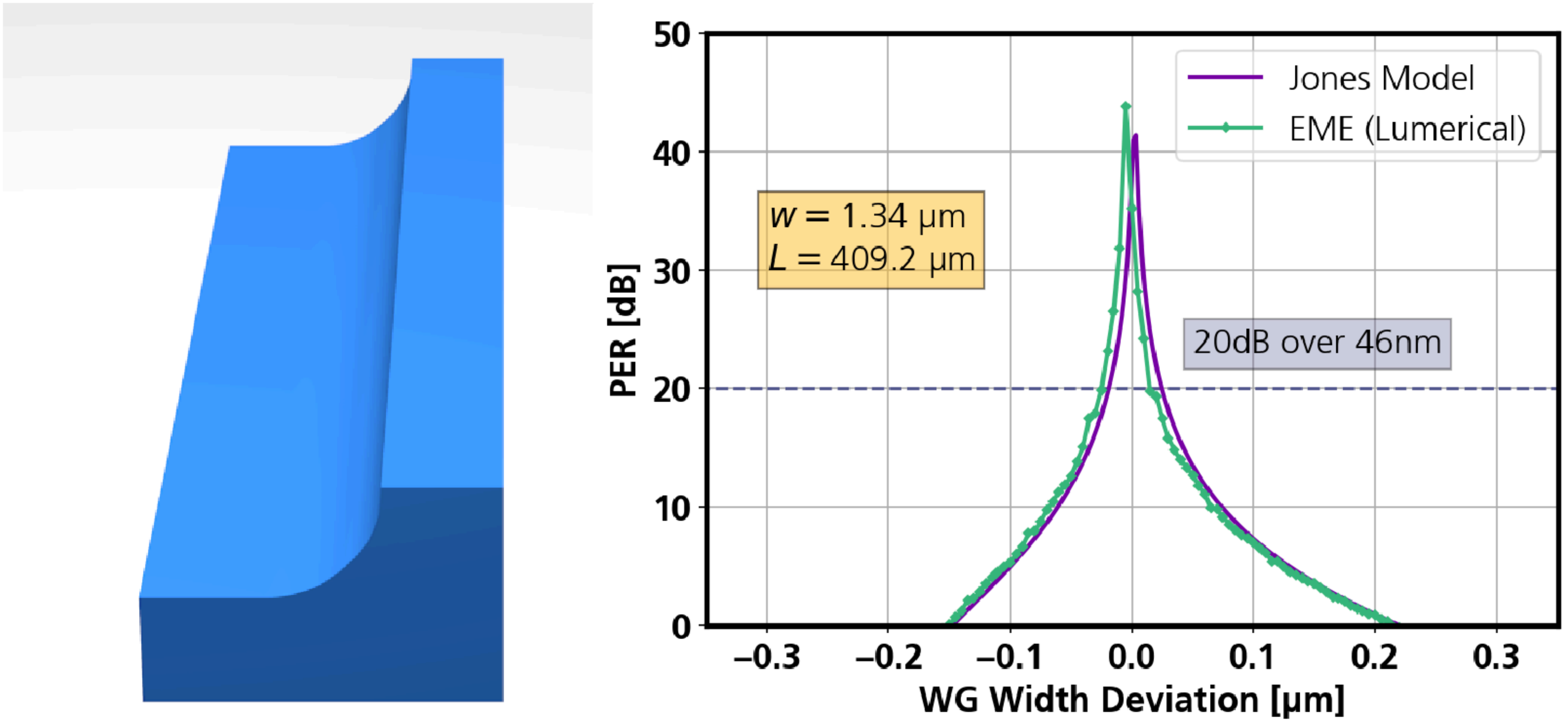
- In the Figure Polarization Extinction Ratio (PER) is shown for a polarization rotator based on an asymmetrical waveguide.

- It is defined as:

$$\mathrm{PER} = \frac{\int |E_y|^2 dxdy}{\int |E_x|^2 dxdy}.$$

Disadvantage:

This type of devices requires high accuracy in fabrication and is usually sensitive to wavelength.

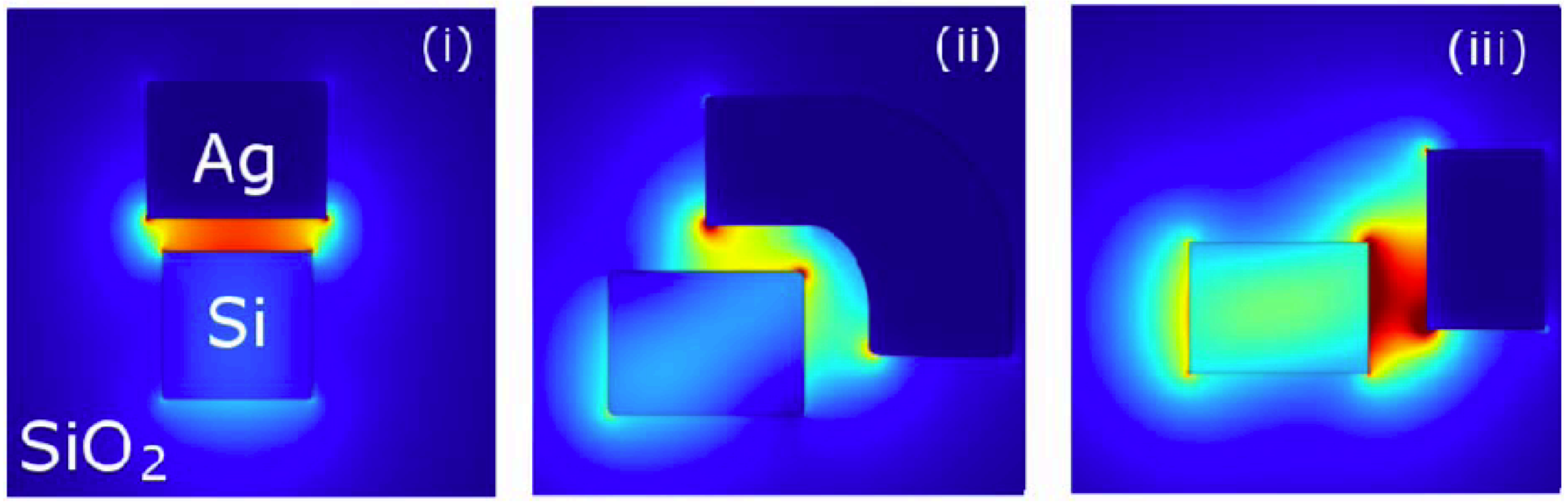


Left: Rendered 3D cartoon of the polarization converter based on the asymmetric waveguide;
Right: dependence of the PER on the waveguide width deviation from the ideal width of 1.34 µm. [1]

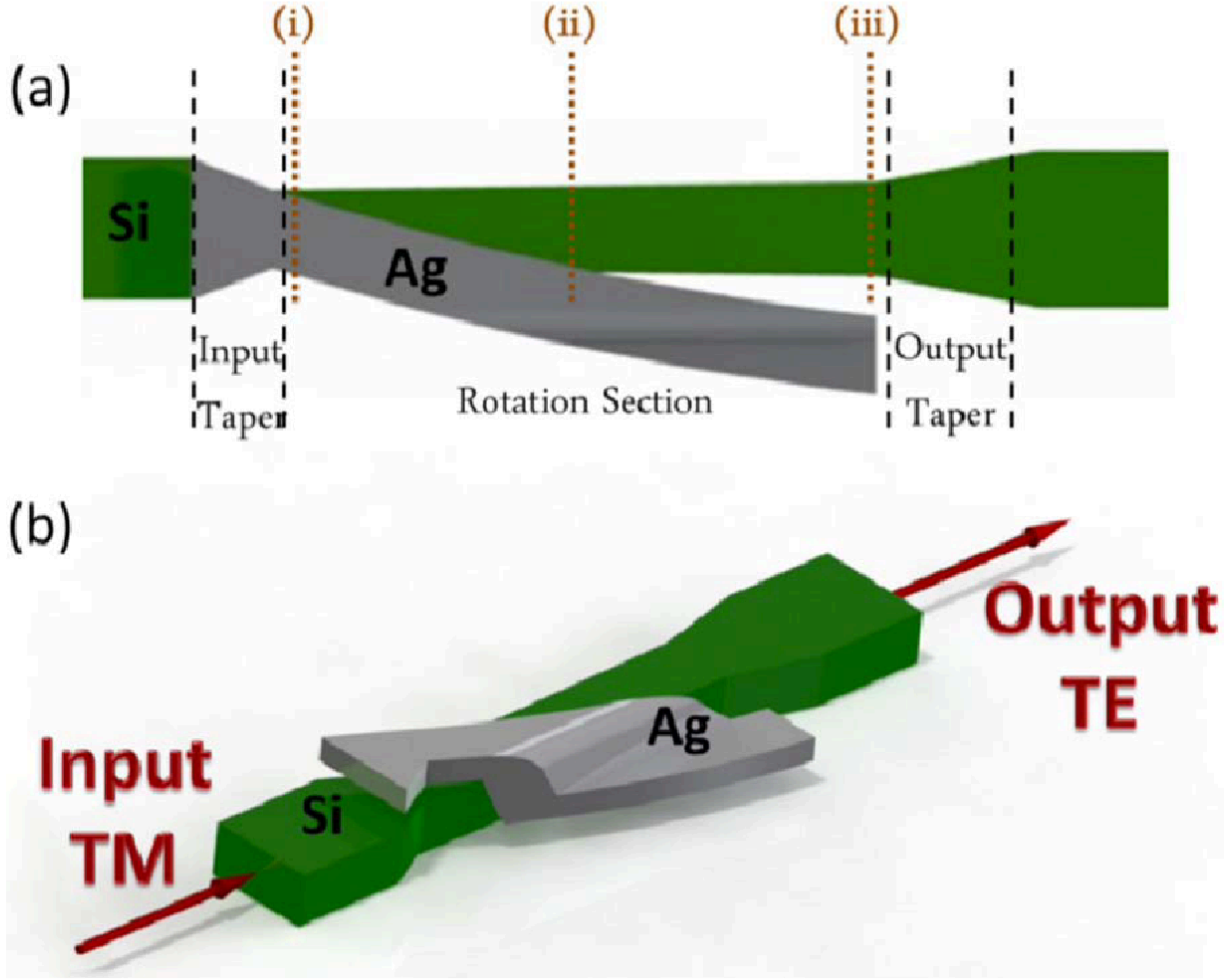
[1] M. F. Baier, Polarization Multiplexed Photonic Integrated Circuits for 100 Gbit/s and Beyond, (2018)

Hybrid Plasmonic Polarization Rotator

- Hybrid Plasmonic (HP) polarization rotators aim to combine advantages of plasmonic schemes (compact size) with low loss operation of dielectric platforms.
- Polarization rotator reported in [1] has ~5 um length, exhibits PER of 14 dB and Insertion loss of 2.1 dB.



Norm of the electric field profiles of the rotated mode along the waveguide. Positions of the profile along the rotator are indicated in the right figure. The positions correspond to (i) input, (ii) middle, and (iii) output of the rotation section.



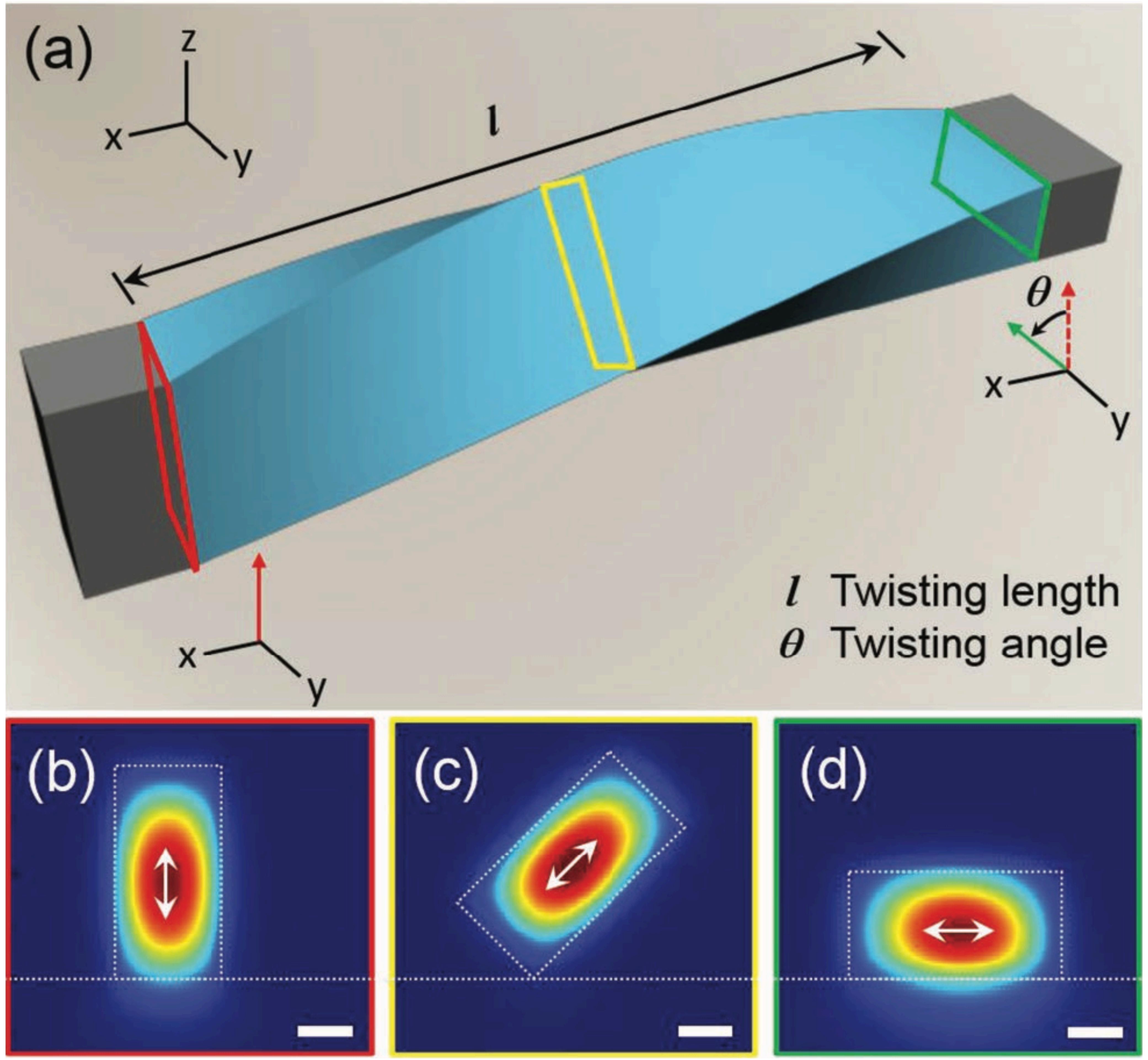
(a) Top view of the polarization rotator, where the black dashed lines separate the three sections: input taper, rotation section, and output taper.
(b) 3D schematics (not to scale) of the rotator. Green corresponds to silicon, grey is silver. The silica is not shown for clarity.

[1] J. Caspers, *et al.*, Compact Hybrid Plasmonic Polarization Rotator, Opt. Lett. **37**, 4615 (2012)

Twisted Waveguides

Polarization Rotator Based on Twisted Waveguides

- Polarization rotation in a twisted waveguide is based on adiabatic mode conversion, where the polarization of the photon changes as the waveguide geometry is slowly engineered.
- Such polarization rotators have a lot of advantages.
- For example, their broad bandwidth benefits the operations involving pulsed lasers.
- Unlike the previously considered rotators, their requirement on fabrication accuracy is relaxed due to the nature of adiabatic mode evolution.



Schematic representation of the twisted waveguide based on adiabatic mode conversion. The simulated electric field distributions at the b) input, c) middle, and d) output of the twisted waveguide, with $\theta = 90°$ and $l = 200$ μm. White arrows in the center denote the polarization of the optical field. Scale bars: 1 μm.

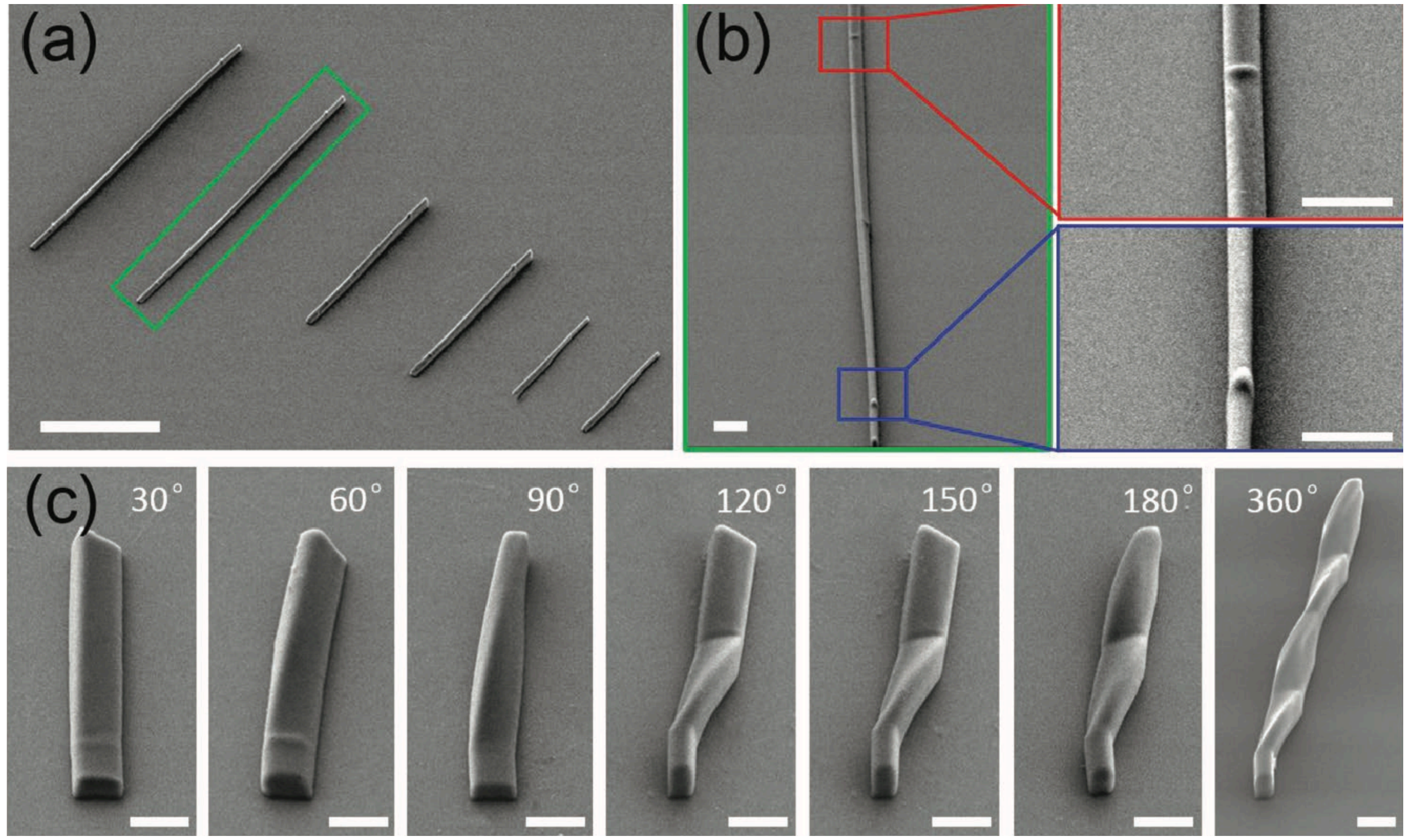
[1] Z. Hou *et al.*, On-Chip Polarization Rotators, Advanced Optical Materials **7**, 1900129 (2019)

Polarization Rotator Based on Twisted Waveguides

- Although the top-down lithography techniques are restricted in their fabrication, people have developed many other techniques that can deal with 3D nanostructures, such as self-assembly and 3D printing.

- Femtosecond direct laser writing (fs-DLW) is one of them, which uses tightly focused ultrashort (fs scale) pulsed laser and changes the optical properties of materials via nonlinear multiphoton absorption.

- Depending on the different materials and the applied laser power, the material properties may change because of polymerization, reduction, bond cleavage, phase change, and ablation.



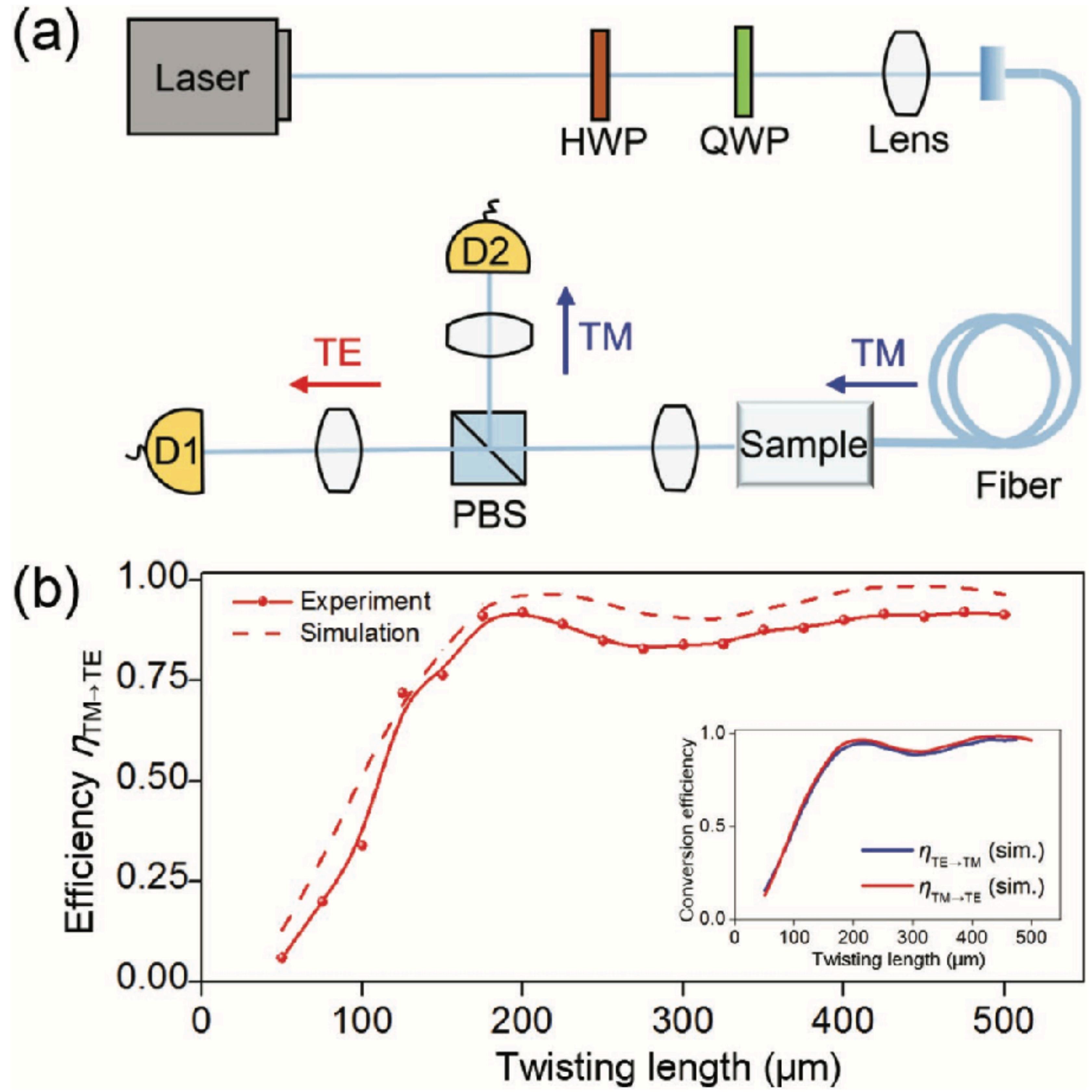
a) SEM image of the polarization rotators with twisting lengths of 150, 100, and 50 μm (from left to right). Scale bar: 100 μm.
b) Enlarged view of the sample selected as green rectangle in panel (a) with twisting length of 150 μm. Insets: zoom-in of the coupling parts. Scale bars: 10 μm.
c) SEM images of twisted segments with different twisting angles. Scale bars: 10 μm.

[1] Z. Hou *et al.*, On-Chip Polarization Rotators, Advanced Optical Materials **7**, 1900129 (2019)

Polarization Rotator Based on Twisted Waveguides

- The experimental setup to characterize the polarization rotators is displayed in the figure. The laser (λ = 1.550 nm or 646 nm) is coupled into the sample with single-mode fiber (SMF).
- Before injected into the sample, the light is tuned as TM polarization by passing through the half-wave plate (HWP) and quarter-wave plate (QWP).
- The output from the sample is collected with lens and directed into two paths with polarization beam splitter (PBS). Two power meters (D1 and D2) are used to record the TE and TM components of waveguide output, respectively.
- Denoting the counts of D1 and D2 as P_{TE} and P_{TM}, we obtain the polarization conversion efficiency expressed as

$$\eta_{\mathrm{TM}\rightarrow\mathrm{TE}} = \frac{P_{\mathrm{TE}}}{P_{\mathrm{TM}} + P_{\mathrm{TE}}}.$$



a) Experimental setup to measure the polarization rotators. HWP: half-wave plate; QWP: quarter-wave plate; PBS: polarization beam splitter; D: detector. b) The polarization conversion efficiency $\eta_{\mathrm{TM}\rightarrow\mathrm{TE}}$ for polarization rotators at λ=1.55 um with different twisting length. Inset: The polarization conversion efficiency with either TM or TE polarization input from the same input port.

[1] Z. Hou *et al.*, On-Chip Polarization Rotators, Advanced Optical Materials **7**, 1900129 (2019)

Description of Light Propagation in a Twisted Waveguide

Simulation Techniques for Twisted Waveguides

Method	Issues
Finite Difference Time Domain (**FDTD**)	➡Need for simulation in 3 dimensions ➡Very Memory Intensive (tens to hundreds GB of RAM per simulation) ➡Long Computation Times ➡Fast prototyping on a personal desktop (laptop) is problematic
Finite Difference Frequency Domain (**FDFD**)	
Finite Element Method (**FEM**)	
Beam Propagation Method (**BPM**)	➡Neglects backward-scattered field ➡Neglects z-derivatives of refractive index ➡Very CPU and RAM efficient ➡Fails to reproduce polarization rotation :(
Eigenmode Expansion Method (**EME**)	➡Calculates both forward and backward fields ➡Efficient for piecewise-uniform structures in propagation direction ➡Requires fine longitudinal discretization for continuously-varying structures ➡Becomes impractical for twisted waveguides

Utilizing “Twisted” Symmetry

- A twisted waveguide with constant twist rate certainly possesses some *symmetry*: its cross-section in every $z = \mathrm{const}$ plane is constant up to rotation and twist rate is also z-independent.

- If we are able to utilize this “twisted” symmetry, we can split variables in Maxwell’s equations and reduce the dimension of the problem.

- To make use of the symmetry, special coordinates can be used.

- In these coordinates we want partial derivative of $n(\mathbf{r})$ in the longitudinal direction to be zero: $\partial_Z n(X, Y, Z)\,|_{X,Y=\mathrm{const}} = 0$.

- Thus, in analogy with the straight waveguide case we can define an eigenmode for a twisted waveguide with $e^{\mathrm{i}\beta Z}$ longitudinal dependence.

Twisted Coordinates

- The coordinates satisfying the requirements can be found:

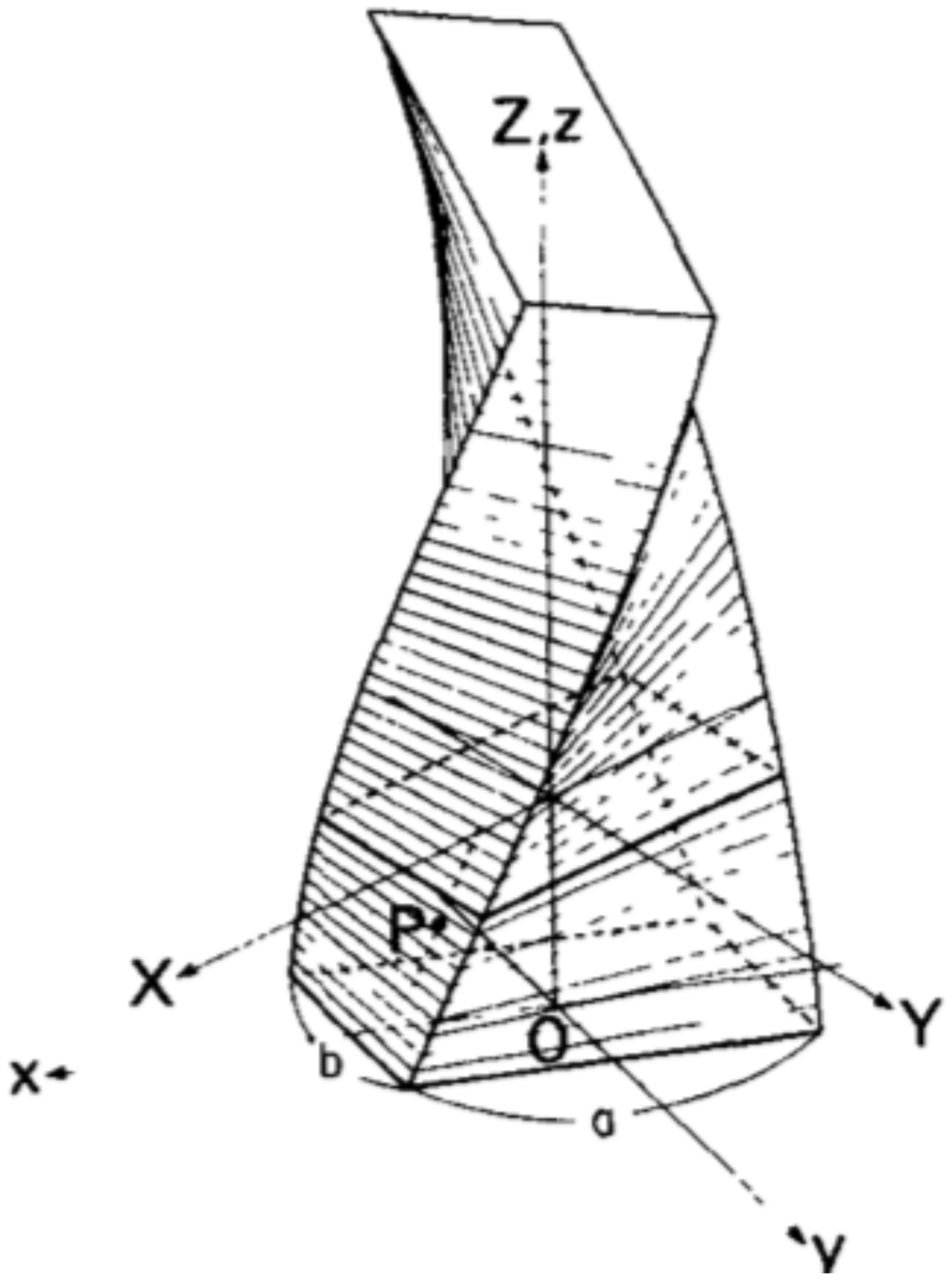
$$X = \cos \alpha z + y \sin \alpha z,$$
$$Y = -x \sin \alpha z + y \cos \alpha z, ,$$
$$Z = z\,.$$

where α [rad/m] is a twist constant.

- We will call them **twisted coordinates**.

- Indeed, in these coordinates

$$\partial_Z n(X, Y, Z) = 0.$$



Twisted rectangular waveguide and coordinate systems [1].
Twisted (X, Y, Z) and Cartesian (x, y, z) coordinates.

[1] H. Yabe and Y. Mushiake, An Analysis of a Hybrid-Mode in a Twisted Rectangular Waveguide, IEEE Trans. Microwave Theory Techn. **32**, 65 (1984)

Twisted Coordinates

$$
\begin{aligned}
X &= \cos \alpha z + y \sin \alpha z, \\
Y &= -\, x \sin \alpha z + y \cos \alpha z, , \\
Z &= z\,.
\end{aligned}
$$

- With this coordinate system two dual sets of base vecto are associated:

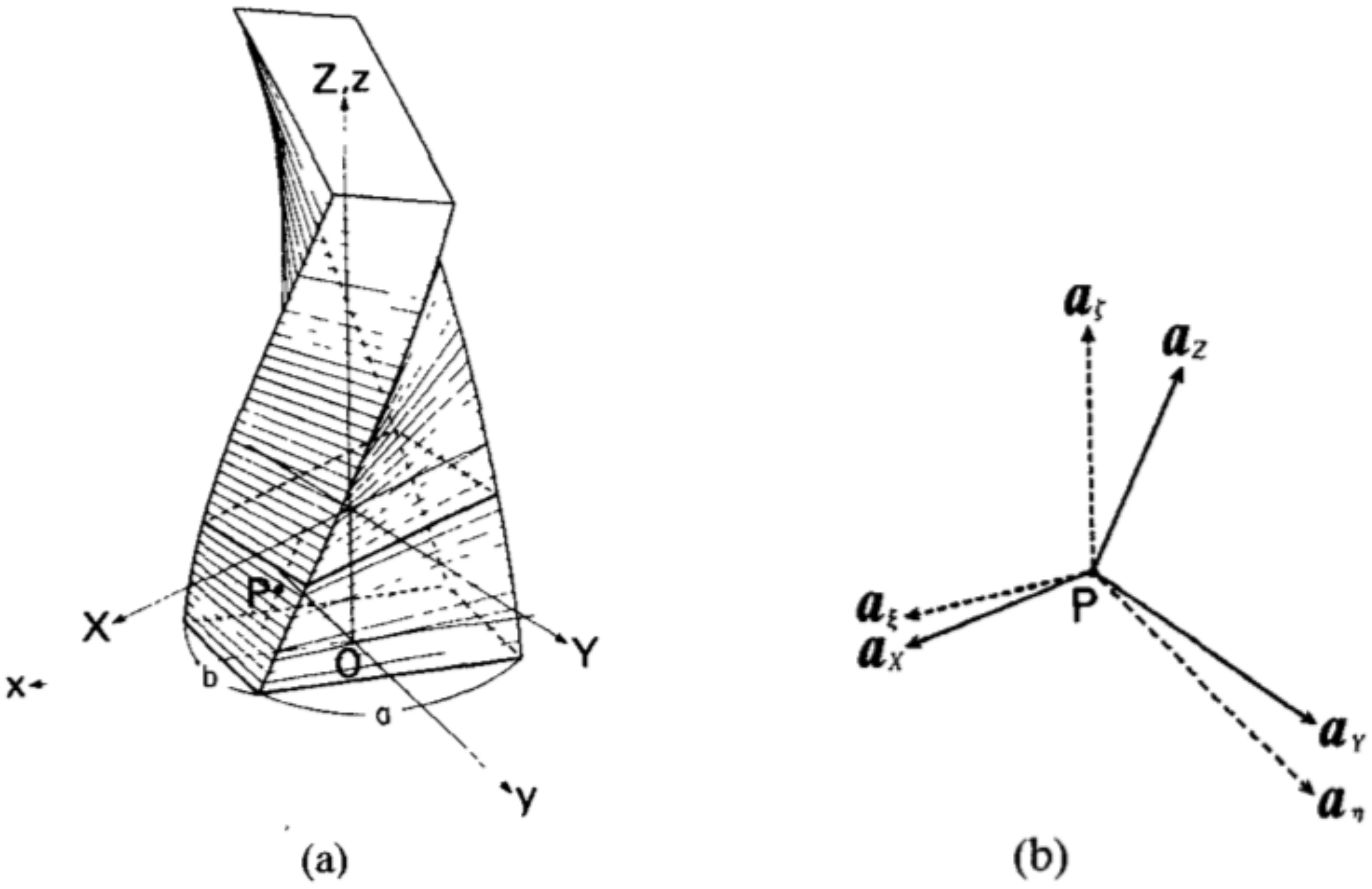
$$
\begin{aligned}
\mathbf{a}_X &= \hat{\mathbf{x}} \cos \alpha z + \hat{\mathbf{y}} \sin \alpha z, \\
\mathbf{a}_Y &= -\, \hat{\mathbf{x}} \sin \alpha z + + \hat{\mathbf{y}} \cos \alpha z, \\
\mathbf{a}_Z &= -\, \alpha y \hat{\mathbf{x}} + \alpha x \hat{\mathbf{y}} + \hat{\mathbf{z}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{a}_\xi &= \mathbf{a}_Y \times \mathbf{a}_Z, \\
\mathbf{a}_\eta &= \mathbf{a}_Z \times \mathbf{a}_X, \\
\mathbf{a}_\zeta &= \mathbf{a}_X \times \mathbf{a}_Y\,.
\end{aligned}
$$

- Any vector can be expanded into both sets of bases:

$$
\mathbf{v} = v_X \mathbf{a}_X + v_Y \mathbf{a}_Y + v_Z \mathbf{a}_Z = v_\xi \mathbf{a}_\xi + v_\eta \mathbf{a}_\eta + v_\zeta \mathbf{a}_\zeta.
$$



Twisted rectangular waveguide and coordinate systems [1].
(a) Twisted (X, Y, Z) and Cartesian (x, y, z) coordinates.
(b) Local oblique base vectors at a point P.

[1] H. Yabe and Y. Mushiake, An Analysis of a Hybrid-Mode in a Twisted Rectangular Waveguide, IEEE Trans. Microwave Theory Techn. **32**, 65 (1984)

Distances in Curvilinear Coordinates and Metric Tensor

- In Cartesian coordinates $x^i = \{x, y, z\}$ an interval between points x^i and $x^i + dx^i$ is expressed as

$$ds^2 = dx^2 + dy^2 + dz^2 = \delta_{ij} dx^i dx^j,$$

where δ_{ij} is the Kronecker delta and Einstein summation rule is assumed.

- For general coordinates, the square of the line element ds^2 is given by an expression quadratic in the coordinate differentials dx^i:

$$ds^2 = g_{ik} dx^i dx^k$$

where g_{ik} is called **metric tensor**.

- Metric tensor expresses **how the distances are measured** in a given coordinate system and it is always *symmetric*.

[1] U. Leonhardt and T. Philbin, Chapter 2 Transformation Optics and the Geometry of Light, in Progress in Optics, Vol. 53 (Elsevier, 2009), pp. 69–152

Distances in Curvilinear Coordinates and Metric Tensor

- Due to invariance of the interval ds^2 we can always relate its expressions in any two coordinate systems:

$$ds^2 = g_{i'k'}dx^{i'}dx^{k'} = g_{ik}dx^i dx^k = g_{ik}J^i_{i'}J^k_{k'}dx^{i'}dx^{k'},$$

where $J^i_{i'}$ is the basis transformation matrix (Jacobian).

- Thus we obtain transformation rule for metric tensor:

$$g_{ik} = J^{i'}_i J^{k'}_k g_{i'k'}.$$

- Forward Jacobian is defined as $J^{i'}_i = \dfrac{\partial x^{i'}}{\partial x^i}$. Jacobian of the transformation from Cartesian to twisted system is

$$J^{i'}_i = \begin{pmatrix} \cos\alpha Z & \sin\alpha Z & \alpha Y \\ -\sin\alpha Z & \cos\alpha Z & -\alpha X \\ 0 & 0 & 1 \end{pmatrix}.$$

- Inverse Jacobian is defined as $J^i_{i'} = \dfrac{\partial x^i}{\partial x^{i'}}$. Jacobian of the transformation from twisted to Cartesian system is

$$J^i_{i'} = \begin{pmatrix} \cos\alpha Z & -\sin\alpha Z & -\alpha(Y\cos\alpha Z + X\sin\alpha Z) \\ \sin\alpha Z & \cos\alpha Z & \alpha(X\cos\alpha Z - Y\sin\alpha Z) \\ 0 & 0 & 1 \end{pmatrix}.$$

[1] U. Leonhardt and T. Philbin, Chapter 2 Transformation Optics and the Geometry of Light, in Progress in Optics, Vol. 53 (Elsevier, 2009), pp. 69–152

Metrics in Twisted Coordinates

- It can be shown that in twisted coordinates interval ds^2 is expressed as

$$ds^2 = dX^2 + dY^2 + \left(1 + \alpha(X^2 + Y^2)\right) dZ^2 + 2\alpha(XdYdZ - YdXdZ),$$

so the metric tensor in twisted coordinates is

$$g_{ik} = \begin{pmatrix} 1 & 0 & -\alpha Y \\ 0 & 1 & \alpha X \\ -\alpha Y & \alpha X & 1 + \alpha^2(X^2 + Y^2) \end{pmatrix}.$$

- **Importantly**, that g_{ik} ***does not depend*** on z.
- As we discussed previously, vectors in twisted coordinates can be expanded in two sets of bases:

$$\mathbf{v} = v_X \mathbf{a}_X + v_Y \mathbf{a}_Y + v_Z \mathbf{a}_Z = v_\xi \mathbf{a}_\xi + v_\eta \mathbf{a}_\eta + v_\zeta \mathbf{a}_\zeta.$$

- We will refer to $v^i = \{v_X, v_Y, v_Z\}$ and $v_i = \{v_\xi, v_\eta, v_\zeta\}$ as contravariant and covariant components, respectively.
- Co- and contravariant components are connected to each other by the metric tensor g_{ik}:

$$v_i = g_{ik} v^k, \; v^i = g^{ik} v_k.$$

Maxwell Equations In Tensor Notation

- When working with curvilinear coordinates it is convenient to express equations in tensor notation.
- Maxwell equations for time-harmonic fields in tensor notation have the following form [1]. Here we use Gauss units:

$$\varepsilon^{ijk}\nabla_k H_j = -\mathrm{i}k_0 n^2 E^i,$$

$$\varepsilon^{ijk}\nabla_k E_j = \mathrm{i}k_0 H^i,$$

$$\nabla_i n^2 E^i = 0,$$

$$\nabla_i H^i = 0,$$

where n is refractive index, ε^{ijk} is fully antisymmetric (Levi-Civita) tensor and ∇_i is the **covariant derivative**.

[1] A. McConnell, *Applications of Tensor Analysis* (Dover Publ, New York, 1957)

Covariant Derivative

- Components of a vector change from point to point not only because the vector change but also because basis vectors change.
- Covariant derivative expresses how the geometric objects (e.g. vectors) change from point to point.
- By definition, covariant derivative of a vector **along a geodesic** line is **zero** because geodesic is defined as the curve everywhere tangent to the parallel-transported vector.
- Also, covariant derivative of the **metric tensor** is **zero**.
- For scalar quantities it is just equal to partial derivative

$$\nabla_i f(\mathbf{x}) = \partial_i f(\mathbf{x}).$$

- Covariant derivatives of contravariant and covariant vectors are calculated as [2]:

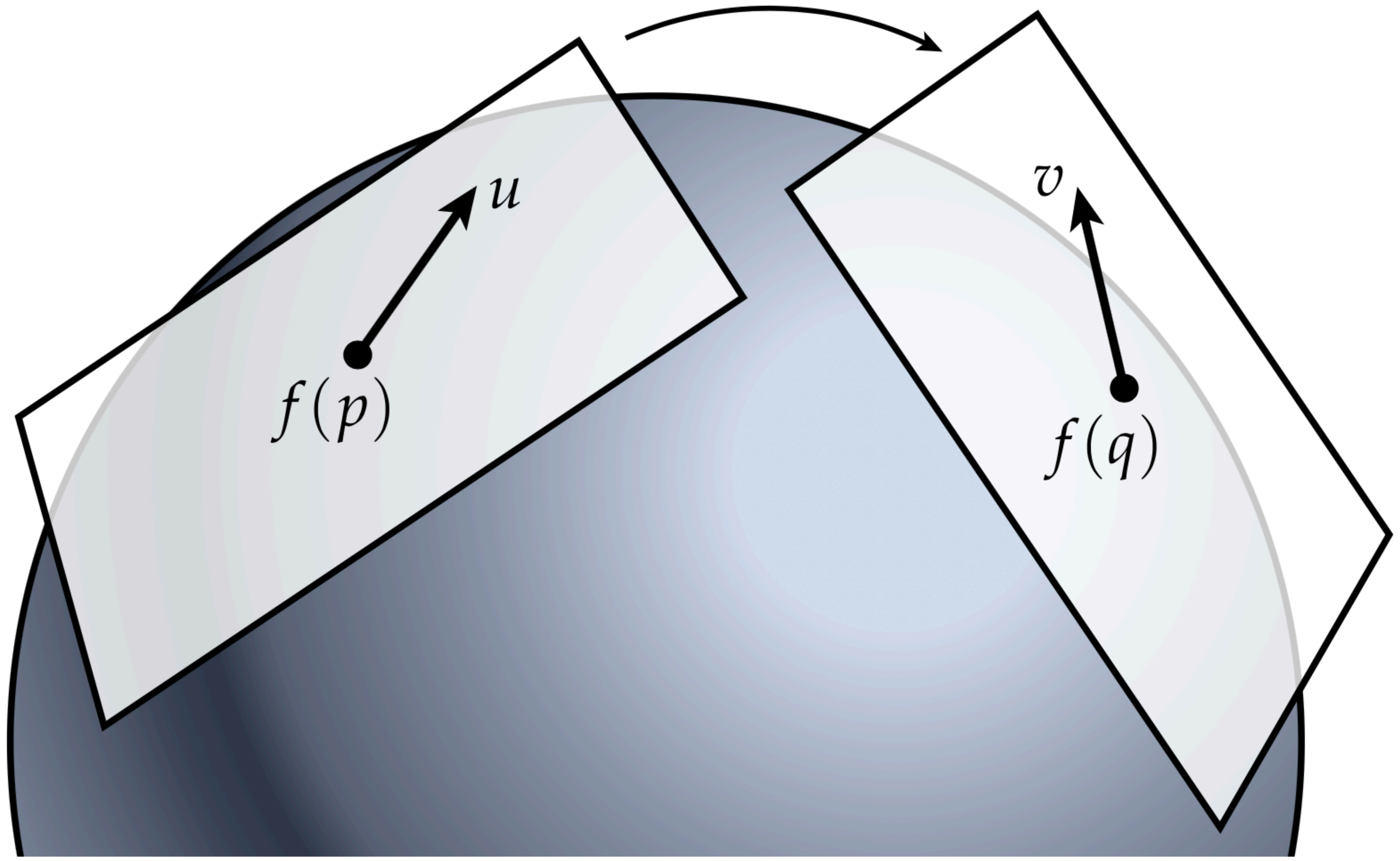
$$\nabla_k v^i = \partial_k v^i + \Gamma^i_{sk} v^s, \nabla_k u_i = \partial_k u_i - \Gamma^s_{ik} u_s,$$

where

$$\Gamma^i_{sk} = \frac{1}{2} g^{is}(\partial_k g_{ls} + \partial_l g_{sk} - \partial_s g_{kl})$$

are *Christoffel symbols*, or, in general, *connection*.

- Once we know the metric tensor in a given basis, we directly obtain the equations in this basis.



Schematics of parallel transport [1]

[1] http://wordpress.discretization.de/geometryprocessingandapplicationsws19/connections-and-parallel-transport/
[2] A. McConnell, *Applications of Tensor Analysis* (Dover Publ, New York, 1957)

Maxwell Equations In Twisted Coordinates

- Again, the main purpose of using twisted coordinates is that in case of twisted waveguide refractive index n **does not depend on** Z **explicitly**:

$$\partial_Z n(X, Y, Z)\,|_{X,Y=\mathrm{const}} = 0.$$

- As we pointed earlier, metric tensor entering Christoffel symbols does not depend on Z as well

$$\partial_Z g_{ik} = 0.$$

- The Christoffel symbols expressed in terms of derivatives of the metric tensor are obviously also Z -independent. Their expressions are:

$$\Gamma^1_{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & -\alpha & -\alpha^2 X \end{pmatrix}, \Gamma^2_{ik} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & -\alpha^2 Y \end{pmatrix}, \Gamma^3_{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- This allows to use **variables splitting** for solving Maxwell equations in a twisted waveguide.

Eigenmodes of a Twisted Waveguide

- In analogy with the straignt case, we can define an eigenmode for a twisted waveguide:

$$E^i(X, Y, Z) = e^i(X, Y)\mathrm{e}^{\mathrm{i}\beta Z},$$
$$H^i(X, Y, Z) = h^i(X, Y)\mathrm{e}^{\mathrm{i}\beta Z}.$$

- Substituting solutions in these form into Maxwell's equations in twisted coordinates we obtain

$$\begin{aligned}
\partial_Y h_Z - \mathrm{i}\beta h_Y - \alpha\left(h_X + \mathrm{i}\beta X h_Z + Y\partial_Y h_X - X\partial_Y h_Y\right) + \alpha^2\left(2Yh_Z + (X^2+Y^2)\partial_Y h_Z\right) &= \mathrm{i}k_0 n^2 e_X \\
-\partial_X h_Z + \mathrm{i}\beta h_X - \alpha\left(h_Y + \mathrm{i}\beta Y h_Z + X\partial_X h_Y - Y\partial_X h_X\right) - \alpha^2\left(2Xh_Z + (X^2+Y^2)\partial_X h_Z\right) &= \mathrm{i}k_0 n^2 e_Y \\
\partial_X h_Y - \partial_Y h_X + \alpha\left(Y\partial_X h_Z + X\partial_Y h_Z + 2h_Z\right) &= \mathrm{i}k_0 n^2 e_Z \\
\partial_Y e_Z - \mathrm{i}\beta e_Y - \alpha\left(e_X + \mathrm{i}\beta X e_Z + Y\partial_Y e_X - X\partial_Y e_Y\right) + \alpha^2\left(2Ye_Z + (X^2+Y^2)\partial_Y e_Z\right) &= -\mathrm{i}k_0 h_X \\
-\partial_X e_Z + \mathrm{i}\beta e_X - \alpha\left(e_Y + \mathrm{i}\beta Y e_Z + X\partial_X e_Y - Y\partial_X e_X\right) - \alpha^2\left(2Xe_Z + (X^2+Y^2)\partial_X e_Z\right) &= -\mathrm{i}k_0 h_Y \\
\partial_X e_Y - \partial_Y e_X + \alpha\left(Y\partial_X e_Z + X\partial_Y e_Z + 2e_Z\right) &= -\mathrm{i}k_0 h_Z
\end{aligned}$$

Perturbative Solutions

- Eigenmode equation for a twisted waveguide can be formally written in operator form

$$\hat{H}\psi = \beta\psi,$$

where $\psi \equiv (e^i, h^i)^T$ and $\hat{H}$ is the differential operator obtained from the Maxwell's equations in twisted coordinates.

- $\hat{H}$ has the form

$$\hat{H} = \hat{H}^{(0)} + \alpha\hat{H}^{(1)} + \alpha^2\hat{H}^{(2)},$$

where $\hat{H}^{(0)}$ is the operator coinciding with the eigenmode equation operator in the absence of twist. Explicit forms of $\hat{H}^{(1,2)}$ are quite cumbersome.

- Assuming that the twist constant α is small (e.g, in sense $\alpha/k_0 \ll 1$) we can expand modal fields and propagation constants in series of α

$$\psi = \psi^{(0)} + \alpha\psi^{(1)} + \alpha^2\psi^{(2)} + \ldots$$

- As was firstly pointed in [1], propagation constant can only contain even powers of α in the expansion, because the twist direction does not affect it, so

$$\beta = \beta^{(0)} + \alpha^2\beta^{(2)} + \ldots$$

[1] L. Lewin, Propagation in Curved and Twisted Waveguides of Rectangular Cross-Section, Proceedings of the IEE - Part B: Radio and Electronic Engineering **102**, 75 (1955)

Perturbative Solutions

- For the unperturbed system corresponding to a straight waveguide

$$\hat{H}^{(0)}\psi_m^{(0)} = \beta_m^{(0)2}\psi_m^{(0)},$$

where $\psi \equiv (e^i, h^i)^T$ and $\hat{H}$ is the differential operator obtained from the Maxwell's equations in twisted coordinates.

- The exact solution is expanded into an orthogonal set of modes of the unperturbed (straight) waveguide modes:

$$\psi_m = \sum_m C_{m\tilde{m}}\psi^{(0)}.$$

- By using orthogonality relations of unperturbed modes we can find the exact equations for the expansion coefficients in terms of matrix elements of perturbation operators

$$\left(\beta_m - \beta_k^{(0)}\right) C_{nk} = \sum_m (\alpha H_{km}^{(1)} + \alpha^2 H_{km}^{(2)})C_{nm},$$

$$H_{km}^{(1,2)} = \langle \psi_k^{(0)} | \hat{H}^{(1,2)} | \psi_m^{(0)} \rangle, \text{ where } \langle \psi_1 | \psi_2 \rangle = \frac{1}{2}\int (\mathbf{e}_1 \times \mathbf{h}_2^*)_z dxdy.$$

- The perturbation theory corresponds to the expansion of values in question into series over α:

$$C_{nk} = C_{nk}^{(0)} + \alpha C_{nk}^{(1)} + \alpha^2 C_{nk}^{(2)} + \dots$$

[1] I. Feranchuk, *et al., Non-Perturbative Description of Quantum Systems* (Springer Berlin Heidelberg, New York, 2015)

"Twisted" Eigenmode Expansion Method

- Provided that we have found the eigenmodes of a twisted waveguide we can formulate a specific *EME method* suitable for twisted waveguides.

- Within this method, the total field in the twisted waveguide is expressed as

$$\mathbf{E}(X, Y, Z) = \sum_m A_m \mathbf{e}_m(X, Y) e^{\mathrm{i}\beta_m Z} + \sum_m B_m \mathbf{e}_{-m}(X, Y) e^{-\mathrm{i}\beta_m Z},$$

where $\mathbf{E} \equiv E^k$, $\mathbf{e} \equiv e^k$ are the fields in twisted coordinates, A_m, B_m are complex amplitudes.

- To find $\mathbf{E}(X, Y, Z)$ we need to:

  A) find the eigenmodes $\mathbf{e}_m$ by solving eigenmode equations in twisted coordinates;

  B) find amplitudes A_m and B_m.

“Twisted” Eigenmode Expansion Method

- Let us assume that a twisted waveguide is facet-coupled to a straight waveguide at $Z = 0$. And a single mode with the field $\tilde{\mathbf{E}} = \tilde{\mathbf{e}} \exp(\mathrm{i}\tilde{\beta}Z)$ is launched into the straight waveguide.
- The continuity of transverse fields requires

$$\tilde{\mathbf{E}}_t(X, Y, 0-) = \mathbf{E}_t(X, Y, 0+),\ \tilde{\mathbf{H}}_t(X, Y, 0-) = \mathbf{H}_t(X, Y, 0+).$$

- Substituting modal expansion for the field in the twisted waveguide we obtain

$$\tilde{\mathbf{e}}_t(X, Y) = \sum_m A_m \mathbf{e}_{m,t}(X, Y) + \sum_m B_m \mathbf{e}_{-m,t}(X, Y),$$

$$\tilde{\mathbf{h}}_t(X, Y) = \sum_m A_m \mathbf{h}_{m,t}(X, Y) + \sum_m B_m \mathbf{h}_{-m,t}(X, Y)\,.$$

- We can expand the modal fields of the twisted waveguide in terms of modes of the straight waveguide:

$$\mathbf{e}_{m,t} = \sum_{\tilde{m}} C_{m\tilde{m}} \tilde{\mathbf{e}}_{\tilde{m},t},\ \mathbf{h}_{m,t} = \sum_{\tilde{m}} C_{m\tilde{m}} \tilde{\mathbf{h}}_{\tilde{m},t}$$

with expansion coefficients obtained by perturbation approach.

- By substituting expansions of the “twisted” modes in terms of “untwisted” ones and applying the orthogonality of the “untwisted” modes $\tilde{\mathbf{e}}_{\tilde{m}}$ we find the amplitudes A_m, B_m.

Citation

Issues

- Definition of eigenmode are possible in terms of co- and contravariant vectors $e_i(X, Y), e^i(X, Y)$ and so with h^i.
- So there are at least 4 possible variants and this is up to us to chose from them. Each variant has its advantages and disadvantages.
- This number is multiplied by several possible ways to formulate the eigenmode equation from Maxwell's equations.
- The perturbational corrections are expressed in terms of cumbersome integro-differential forms of unperturbed fields.
- Radiation modes must also contribute to the expansions of twisted modes. But it is a challenge to take radiation modes into account when finding eigenmodes numerically.

Summary

- Polarization diversity schemes are required to obtain high data rates in FO communication systems combined with SOI integrated devices.
- Polarization rotators are key components in these schemes.
- Twisted waveguide is a promising platform for realization of polarization rotators.
- Although they can be modeled with full-wave 3D numerical approaches, there is demand of a method making use of the geometry.
- The idea of “Twisted EME” is proposed. It requires knowing of the modes of the twisted waveguide i.e. longitudinally-exponential solutions with constant profiles in rotating cross-sectional plane.
- The modes of twisted waveguides can be found with perturbation theory as expansion in terms of modes of a straight waveguide.